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Simple calculation of the critical mass for highly enriched uranium and plutonium-239

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The correct calculation of and values for the critical mass of uranium or plutonium necessary for a nuclear fission weapon have long been understood and publicly available. The calculation requires solving the radial component in spherical coordinates of a diffusion equation with a source term, so is beyond the reach of most public policy and many first-year college physics students. Yet it is important for the basic physical ideas behind the calculation to be understood by those without calculus who are nonetheless interested in international security, arms control, or nuclear non-proliferation. This article estimates the critical mass in an intuitive way that requires only algebra. © 2014 American Association of Physics Teachers.

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I. INTRODUCTION

Fissile materials are materials that can sustain an explosive fission chain reaction.¹ A “threshold bare critical mass” M_c (henceforth just “critical mass”) is the minimum mass of fissile material that is needed for that chain reaction. At the threshold critical mass, neutron production by fission within a volume just balances neutron loss through the volume’s surface, and the number density of neutrons is constant in time. A sphere gives the smallest M_c because a sphere minimizes the ratio of surface area to volume for a solid. “Bare” critical mass indicates that there is no neutron reflector; such a component reflects neutrons that would otherwise escape from the sphere back into its interior.

The “critical radius” R_c is the radius of the sphere of fissile material corresponding to M_c . Real nuclear weapons may be expected to employ neutron reflectors and implosive spherical compression, both of which serve to reduce the amount of fissile material below that of the bare critical mass.¹ Nevertheless, the value of M_c is a useful benchmark for the material requirements for a nuclear weapons program—for example, the rapidity with which a potential nuclear proliferator could produce a bomb’s worth of highly enriched uranium (HEU) or plutonium-239 (Pu) from gas centrifuges or nuclear reactors, respectively. (Weapons grade plutonium produced in a plutonium production reactor is preferable for military purposes to power-reactor plutonium, but the latter may also be used to produce a weapon, albeit with greater difficulty.²) The value of the critical mass provides the fundamental context for arms control efforts to constrain or reduce fissile material stocks below a certain number of equivalent warheads. A basic understanding of the derivation of the critical mass is therefore an important underpinning to key aspects of contemporary international security.

The standard derivation of the critical mass involves solving a time-dependent diffusion equation with a source term, and may be found in Serber’s long-declassified *Los Alamos Primer*³ and in Reed’s *The Physics of the Manhattan Project*.⁴ At the threshold critical mass, the time dependence disappears, and one is left with having to solve the radial component in spherical coordinates of a diffusion equation

with a source term. This or simplified versions may also be found in a number of journal articles.^{5–9}

The ready availability of the derivation and the value of the critical mass for HEU and Pu assures that further publications in this realm cannot provide any significant information to states or terrorists pursuing nuclear weapons. However, a derivation accessible to those without calculus could help ensure that the next generation of students interested in international security, arms control, or nuclear non-proliferation has an intuitive quantitative understanding of the critical mass needed for a nuclear explosion. We provide such a derivation here, in the hopes of capturing the key physical ideas without doing too much damage to the underlying physics through various simplifications.

II. A SIMPLE MODEL

Hafemeister¹⁰ suggests following the simple physical picture of Serber³ to derive R_c by equating neutron production rate P_N due to nuclear fission within a spherical volume to neutron loss rate L_N through the boundary of that volume. Hafemeister provides a straightforward estimate of P_N within a sphere of radius R . Take the typical velocity v of a neutron produced by a nuclear fission to be that corresponding to its kinetic energy (≈ 2 MeV). This neutron has a mean free path between fission events $\lambda_f = 1/n\sigma_f$, where σ_f is the fission cross section and n is the number density of fissile nuclei.¹¹ The nuclear generation lifetime $\tau = \lambda_f v \sim 10^{-8}$ s is the corresponding time between fissions. In each fission, an average number ν of neutrons is produced. Let N be the free neutron number density in the sphere. (In reality, of course, N is a function of radial distance within the sphere,^{3,4} but it is interesting to see how far one can go with a much simpler assumption.) Then P_N is given by the total number of neutrons in the spherical sample, $N(4/3)\pi R^3$, times the net number of neutrons produced per neutron in a nuclear generation lifetime, $(\nu - 1)/\tau$, where the term $\nu - 1$ takes into account the loss of the initiating neutron in each fission event. We therefore have

$$P_N = N \left(\frac{4}{3} \pi R^3 \right) \frac{\nu - 1}{\tau}. \quad (1)$$

The neutron loss rate L_N is the trickier part of the estimate. (Even Serber's Los Alamos lectures had trouble setting up the condition at the loss boundary.³) For his simple model Hafemeister puts

$$L_N = 4\pi R^2 N v, \quad (2)$$

saying that this represents the “neutron loss rate through an area” but does not explain further. However, this expression may be shown to result from an assumption that in a given nuclear generation lifetime all neutrons within a distance λ_f of the outer boundary of the sphere of fissile material escape the sphere, in an approximation where $\lambda_f \ll R$. That is, the number of neutrons in the outermost spherical shell of thickness λ_f is just $[(4/3)\pi R^3 - (4/3)\pi(R - \lambda_f)^3]N \approx 4\pi R^2 \lambda_f N$, with the last approximation holding provided $\lambda_f/R \ll 1$. If these neutrons escape in time τ , Eq. (2) follows because $\lambda_f/\tau = v$. The critical radius $R = R_c$ then results from putting $P_N = L_N$:

$$R_c = \frac{3v\tau}{(\nu - 1)} = \frac{3}{(\nu - 1)} \lambda_f. \quad (3)$$

There are several problems with this simple model. Clearly, not all neutrons within λ_f of the edge will escape, since most will not have a purely radial velocity. Moreover, as Reed⁶ has emphasized, the cross section for neutron scattering in both HEU and Pu is larger than the cross section for fission, so that scattering cannot be neglected even in a lowest-order estimate.

Solving for numerical values emphasizes the problems.¹² For HEU, $\nu = 2.64$ and $\lambda_f = 16.9$ cm, so Eq. (3) gives $R_c = 31$ cm, whereas the correct value is 8.4 cm. Because in reality $\lambda_f > R_c$ for HEU, the implicit assumption $\lambda_f \ll R_c$ underlying Eq. (2) cannot hold. Forcing that assumption in setting up the model necessarily leads, for a self-consistent model, to a value of R_c larger than λ_f , and therefore much larger than the correct value.

III. AN IMPROVED SIMPLE MODEL

Our goal is to treat the estimate for P_N and L_N more carefully so that a self consistent and more accurate value of R_c may result even for a simple model. A key step is to take account of neutron elastic scattering. For both HEU and Pu, the neutron scattering cross section σ_s is several times larger than σ_f , so that the mean free path λ_s between scattering events is several times smaller than λ_f . Over the course of a nuclear generation lifetime τ , a neutron will therefore scatter $\eta = \lambda_f/\lambda_s$ times. We take the distance d_η traveled by a neutron that scatters η times with step size λ_s to be given roughly by the usual random-walk result,^{13,14} $d_\eta \approx \sqrt{\eta} \lambda_s = \lambda_f/\sqrt{\eta}$. Of the neutrons that do have an outward-directed radial velocity, d_η will typically not be oriented in a purely radial direction. In Cartesian coordinates with x lying in the radial direction, the radial distance traveled will be given by $d_\eta^2 = x^2 + y^2 + z^2$, so that typically $d_\eta^2 = 3x^2$ or $x = (1/\sqrt{3})d_\eta$ since there is no preferred direction. Therefore, the radial distance traveled by a typical scattering neutron in time τ is

$$x = \frac{\lambda_f}{\sqrt{3\eta}}. \quad (4)$$

Roughly half these trajectories will be oriented outward toward the boundary of the fissile material, with the other half oriented inward.

We now describe the simple picture underlying our model, building on Hafemeister's approach. We divide our fissile spherical mass into concentric layers. The interior layer is a sphere of radius $R - x$. The outer layer is a spherical shell of thickness x with neutron number density N . In a time τ , half of these outer-layer neutrons—those whose net motion is radially outward—escape the shell and cause no fissions. The other half, whose net motion is radially inward, contribute to maintaining the constant number density N of the inner sphere. Each neutron present within this inner sphere causes a fission event in time τ .

We therefore have, for neutron production,

$$P_N = N \frac{4}{3} \pi (R - x)^3 \frac{\nu - 1}{\tau}. \quad (5)$$

Meanwhile, neutron loss from the outer layer during this same time period is given by

$$L_N = \frac{N 4}{2} \pi \left[\frac{R^3 - (R - x)^3}{\tau} \right]. \quad (6)$$

Equating neutron production and loss (at which point $R \equiv R_c$) gives $R_c^3 = (2\nu - 1)(R_c - x)^3$, or

$$R_c = \frac{1}{\sqrt{3\eta}} \frac{(2\nu - 1)^{1/3}}{(2\nu - 1)^{1/3} - 1} \lambda_f, \quad (7)$$

where we have used Eq. (4). Equation (7) may be contrasted with Eq. (3); the obvious difference is that the coefficient of λ_f in Eq. (7) is smaller than 1, whereas in Eq. (3) it is larger.

IV. IMPROVING THE IMPROVED MODEL

We now evaluate the critical radius given by Eq. (7) for HEU and plutonium (Pu-239).¹² In the case of HEU, we have $\lambda_s = 4.57$ cm so that $\eta = \lambda_f/\lambda_s = 3.7$, $x = 0.30$ $\lambda_f = 5.1$ cm, and $R_c = 13$ cm. The correct value is 8.4 cm, so our estimate is high but correctly places the value as a sphere about a decimeter in radius.

For Pu-239, we have $\nu = 3.17$, $\lambda_s = 5.79$ cm, $\lambda_f = 14.14$ cm, so that $\eta = 2.44$, $x = 0.37 \lambda_f = 5.2$ cm, and $R_c = 12$ cm. This is smaller than the value for HEU, as it should be, but nearly twice the correct value of 6.3 cm. Of course, these errors in R_c are magnified when calculating critical masses; our overestimates by factors of 1.5 and 1.9 for R_c become overestimates by factors of 3.4 and 6.9 for M_c for HEU and Pu-239, respectively.

It is useful to ask why our simple model overestimates R_c . Presumably this is due to overestimating neutron loss relative to neutron production. One obvious contributing factor is the treatment of N as a constant with radius r . But physically, for diffusion leading to loss through a spherical boundary, one expects $N = N(r)$ to fall as one moves from the center to the periphery of the sphere. Neutron escape comes from those radii where the neutron density is lowest, so we overestimate neutron loss with a simple model that takes N to be spatially constant. This in turn requires more fissile material to balance the loss, leading to an overestimate of R_c .

We improve our estimates of M_c by incorporating a more realistic neutron density variation into our model, without rendering the model so complex as to defeat its virtue of simplicity. Quantitative knowledge of $N(r)$ derives from solving

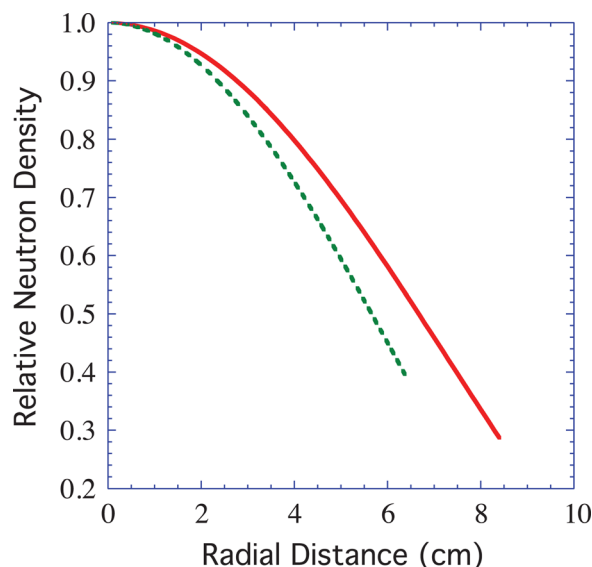


Fig. 1. Relative neutron density as a function of radius in bare critical spheres of HEU (solid curve; critical radius $R_c = 8.4$ cm) and Pu-239 (dashed curve; $R_c = 6.3$ cm).

the radial component of the appropriate diffusion equation, so incorporating results for $N(r)$ involves a certain amount of “cheating.” Full diffusion equation treatments of the problem find that neutron density $N(r)$ scales like⁴

$$N(r) \sim \frac{\sin(r/d)}{r/d}, \quad (8)$$

where

$$d = \left[\frac{\lambda_f \lambda_t}{3(\nu - 1)} \right]^{1/2} \quad (9)$$

and we leave out a normalization constant in Eq. (8). Here λ_t is the neutron total mean free path, equal to 3.60 cm and 4.11 cm for HEU and Pu-239, respectively, and thus giving $d = 3.52$ cm and $d = 2.99$ cm for the two cases. Using these values for d , we plot $N(r)$ [Eq. (8)] for HEU and Pu-239 in Fig. 1.

Fig. 1 shows, rather dramatically, how N falls off with r , contrary to the assumption of our model. An improved approximation would be to define $N(r) = N$ in the innermost sphere of the critical mass and use Fig. 1 to approximate the relative value of $N(r)$ in the outer layer from which neutrons escape. Then, in this “improved improved model,” we would replace N by N/α in Eq. (6), where α is the ratio of N in the innermost sphere to that in the outer spherical layer. With this substitution, Eq. (7) becomes

$$R_c = \frac{1}{\sqrt{3\eta}} \frac{[2\alpha(\nu - 1) + 1]^{1/3}}{[2\alpha(\nu - 1) + 1]^{1/3} - 1} \lambda_f. \quad (10)$$

Recalling that the width of the outer layer is $x = 5.1$ cm and 5.2 cm for HEU and Pu, and recognizing that the total number of neutrons in a sphere or spherical shell with constant neutron density is dominated by those at the largest radial values (since the volume of a shell goes as r^2), Fig. 1

suggests choosing $\alpha = 3$ and 2.5 for HEU and Pu, respectively. With these choices, we find $R_c = 9.3$ cm for both HEU and Pu. These overestimates, by factors of 1.1 and 1.5 for R_c , lead to overestimates in M_c by factors of 1.3 and 3.2 for HEU and Pu, respectively.

V. CONCLUSION

The approach to calculating the critical radius of a fissile isotope described here relies on a simple physical model in which the production of neutrons in the fissile material volume is balanced by the loss of those sufficiently close to the boundary of that volume and moving in the right direction to escape. Critical masses then follow via the densities for HEU or Pu. Our hope is that the intuitive nature of this calculation, and its use of only elementary algebra, will make the origin of the disturbingly small (from a weapons proliferation point of view) critical masses for HEU and Pu accessible to a larger number of students or professionals who wish to be involved in international security, arms control, or nuclear non-proliferation issues.

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¹International Panel on Fissile Materials, *Global Fissile Material Report 2013*, Appendix: Fissile Materials and Nuclear Weapons, <<http://fissile-materials.org/library/gfmr13.pdf>>.

²J. C. Mark, F. von Hippel, and E. Lyman, “Explosive properties of reactor-grade plutonium,” *Sci. Glob. Sec.* **17**, 170–185 (2009).

³R. Serber, *The Los Alamos Primer: The First Lectures on How to Build an Atomic Bomb* (Univ. Calif. Press, Berkeley, 1992).

⁴B. C. Reed, *The Physics of the Manhattan Project*, 2nd ed. (Springer, Heidelberg, 2011).

⁵E. Derrington, “Estimate of the critical mass of a fissionable isotope,” *Am. J. Phys.* **58**, 363–364 (1990).

⁶B. C. Reed, “Estimating the critical mass of a fissionable isotope,” *J. Chem. Educ.* **73**, 162–164 (1996).

⁷J. Bernstein, “Heisenberg and the critical mass,” *Am. J. Phys.* **70**, 911–916 (2002).

⁸B. C. Reed, “Guest Comment: A Simple Model for Determining the Critical Mass of a Fissile Nuclide,” *SPS Observer* **38**(4), 10–14 (2006); available at http://www.spsobserver.org/2006/observer_W06.pdf.

⁹B. C. Reed, “Arthur Compton’s 1941 report on explosive fission of U-235: A look at the physics,” *Am. J. Phys.* **75**, 1065–1072 (2007).

¹⁰D. Hafemeister, *Physics of Societal Issues: Calculations on National Security, Environment, and Energy* (Springer, New York, 2007).

¹¹This definition of the mean free path λ_f is strictly only valid in the limit of an object much larger than λ_f ; it therefore represents another imperfect approximation in these calculations (see Ref. 4, Sec. 2.1).

¹²All data values used in our calculations are taken from Ref. 4, Table 2.1.

¹³M. Harwit, *Astrophysical Concepts* (Wiley, New York, 1982), Chap. 4.

¹⁴Here we make the admittedly incorrect assumption that the elastic neutron scattering is isotropic, but this assumption underlies the diffusion equation approach as well, and seems unavoidable since “much of the physics of this area remains classified or at least not easily accessible...” (Ref. 4, p. 46).