The I Theory of Money*

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Abstract

A theory of money needs a proper place for financial intermediaries. Intermediaries create inside money and their ability to take risks determines the money multiplier. In downturns, intermediaries shrink their lending activity and fire-sell assets. Moreover, they create less inside money, exactly at a time when the demand for money rises. The resulting Fisher disinflation hurts intermediaries and other borrowers. The initial shock is amplified, volatility spikes and risk premia rise. Monetary policy is redistributive. An accommodative monetary policy in downturns, focused on the assets held by constrained agents, recapitalizes balance sheet-impaired sectors and hence mitigates these destabilizing adverse feedback effects. A monetary policy rule that accommodates negative shocks and tightens after positive shocks provides ex-ante insurance, mitigates financial frictions, reduces endogenous risk and risk premia, but also creates moral hazard.

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1 Introduction

A theory of money needs a proper place for financial intermediaries. Financial institutions are able to create money – when they extend loans to businesses and home buyers, they credit the borrowers with deposits and so create inside money. The amount of money created by financial intermediaries depends crucially on the health of the banking system and the presence of profitable investment opportunities. This paper proposes a theory of money and provides a framework for analyzing the interaction between price stability and financial stability. It therefore provides a unified way of thinking about monetary and macroprudential policy.

Intermediaries serve three roles. First, intermediaries monitor end-borrowers. Second, they diversify by extending loans to and investing in many businesses projects and home buyers. Third, they are active in maturity and liquidity transformation as they issue liquid short-term (inside) money and invest in illiquid long-term assets. Intermediation involves taking on some risk. Hence, a negative shock to end borrowers also hits levered intermediary balance sheets. Intermediaries’ individually optimal response to an adverse shock is to shrink their balance sheet. They lend less and accept fewer deposits. As a consequence, the amount of inside money in the economy shrinks. At the same time, because idiosyncratic risk is less well diversified, demand for money increases. Together, both effects lead to increase in the value of outside money, i.e. disinflation occurs.

The disinflationary spiral in our model can be understood through two polar cases. In one polar case the financial sector is undercapitalized and cannot perform its functions. As the intermediation sector does not create any inside money, money supply is scarce and the value of money is high. Households have a desire to hold money which, unlike the household’s own risky individual project, is subject only to aggregate, not idiosyncratic, risk. The value of safe money is high – indeed, as in Samuelson (1958) and Bewley (1980) it is a bubble. In the opposite polar case, intermediaries are well capitalized and so well-equipped to mitigate financial frictions. They are able to exploit diversification benefits by investing across many different projects. Intermediaries also create short-term (inside) money and hence the money multiplier is high. At the same time, since households can offload some of their idiosyncratic risks to the intermediary sector, their demand for money is low. Hence the value of money is low in this polar case.

As intermediaries are exposed to end-borrowers’ risk, an adverse shock also lowers the financial sector’s risk bearing capacity. It moves the economy closer to the first polar regime.
with high value of money. In other words, a negative productivity shock leads to disinflation à la Fisher (1933). Financial institutions are hit on both sides of their balance sheets. On the asset side, they are exposed to productivity shocks of their end-borrowers. End-borrowers’ fire sales depress the price of physical capital and liquidity spirals further erode intermediaries’ net worth (as shown in Brunnermeier and Sannikov (2014)). On the liabilities side, they are hurt by the Fisher disinflation. As intermediaries cut their lending and create less inside money, money demand rises and the real value of their nominal liabilities expands. The Fisher disinflation spiral amplifies the initial shock and the asset liquidity spiral even further. Overall, the economy’s capacity to diversify idiosyncratic risk moves around endogenously.

Monetary policy can work against the adverse feedback loops that precipitate crises, by affecting the prices of assets held by constrained agents and redistributing wealth. That is, monetary policy works through wealth/income effects, unlike conventional New Keynesian models in which monetary policy gains traction by changing intertemporal incentives – a substitution effect. Specifically, in our model, monetary policy softens the blow of negative shocks and helps to maintain the intermediary sector’s capacity to diversify idiosyncratic risk. Thus, it reduces endogenous (self-generated) risk and overall risk premia. Monetary policy is redistributive, but it is not a zero-sum game – it can actually improve welfare.

Simple interest rate cuts in downturns improve economic outcomes only if they boost prices of assets, such as long-term government bonds, that are held by constrained sectors. Wealth redistribution towards the constrained sector leads to a rise in economic activity and an increase in the price of physical capital. As the constrained intermediary sector recovers, it creates more (inside) money and reverses the disinflationary pressure. The appreciation of long-term bonds also mitigates money demand, as long-term bonds can be used as a store of value as well. As banks are recapitalized, they are able to take on more idiosyncratic household risks, so economy-wide diversification of risk improves and the overall economy becomes, somewhat paradoxically, safer. In a sense, this like the Keynesian savings paradox (less individual saving means more aggregate saving), but now applied to risk. Importantly, monetary policy also affects risk premia. As interest rate cuts affect the equilibrium allocations, they also affect the long-term real interest rate as documented by Hanson and Stein (2014) and term premia and credit spread as documented by Gertler and Karadi (2014). From an ex-ante perspective long-term bonds provide intermediaries with a hedge against losses due to negative macro shocks, appropriate monetary policy rule can serve as an insurance mechanism against adverse events.
Like any insurance, “stealth recapitalization” of the financial system through monetary policy can potentially create a moral hazard problem. However, moral hazard from monetary policy is less severe than that associated with explicit bailouts of failing institutions. The reason is that monetary policy is a crude redistributive tool that helps the strong institutions more than the weak. The cautious institutions that bought long-term bonds as a hedge against downturns benefit more from interest rate cuts than the opportunistic institutions that increased leverage to take on more risk. In contrast, ex-post bailouts of the weakest institutions create strong risk-taking incentives ex-ante.

To compare various alternative monetary policy rules we develop a full welfare analysis for our heterogeneous agent model. One might think that a monetary policy rule that fully removes endogenous risk is the optimal one. However, this is not the case due to pecuniary externalities – everybody takes the price of physical capital and the price of money as given. Our analysis shows that investment is excessive and drives up the price of capital beyond the optimal one, lowering the return of capital. In other words, households take on too much idiosyncratic risk. A monetary policy that fully removes endogenous risk only partially completes markets, and so in particular need not be – and in this model is not – welfare-improving, as in the famous Hart (1975) example. A better policy removes some, but not all endogenous risk, striking an optimal balance between removing endogenous risk and leaning against pecuniary externalities. Finally, we show that combining monetary policy with macroprudential policy measures that limit individual households’ undiversifiable risk-taking significantly increases welfare.

**Related Literature.** Our approach differs in important ways from both the prominent New Keynesian approach but also from the monetarist approach. The New Keynesian approach emphasizes the interest rate channel. It stresses the role of money as unit of account and price and wage rigidities are the key frictions. Price stickiness implies that a lowering nominal interest rates also lowers the real interest rate. Households bring consumption forward and investment projects become more profitable. Within the class of New Keynesian models, Christiano, Moto and Rostagno (2003) is closest to our analysis as it studies the disinflationary spiral during the Great Depression. More recently, Cúrdia and Woodford (2010) introduced financial frictions in the new Keynesian framework.

In contrast, our I Theory stresses the role of money as store of value and the redistributive channel of monetary policy. Financial frictions are the key friction. Prices are fully flexible. This monetary transmission channel works primarily through capital gains, as in the asset pricing channel promoted by Tobin (1969) and Brunner and Meltzer (1972). As assets are
not held symmetrically in our setting, monetary policy redistributes wealth and thereby mitigates debt overhang problems. In other words, instead of emphasizing the substitution effect of interest rate changes, the I Theory stresses wealth/income effects of interest rate changes.

Like in monetarism (see e.g. Friedman and Schwartz (1963)), an endogenous reduction of money multiplier (given a fixed monetary base) leads to disinflation in our setting. While inside and outside money have identical return and risk profiles (and so are perfect substitutes in the eyes of an individual investor), they are not the same for the economy as a whole. Inside money serves a special function: By creating inside money, intermediaries are able to diversify risks and foster economic growth. Hence, in our setting monetary intervention should aim to recapitalize undercapitalized borrowers rather than simply increase the money supply across the board. A key difference to our approach is that we focus more on the role of money as a store of value instead of the transaction role of money. The latter plays an important role in the “new monetarist economics” as outlined in Williamson and Wright (2011) and references therein.

Instead of the “money view” our approach is closer in spirit to the “credit view” à la Gurley and Shaw (1955), Patinkin (1965), Tobin (1969, 1970), Bernanke (1983) Bernanke and Blinder (1988) and Bernanke, Gertler and Gilchrist (1999). As in Samuelson (1958) and Bewley (1980), money is essential in our model in the sense of Hahn (1973). In Samuelson households cannot borrow from future not yet born generations. In Bewley (1980) and Scheinkman and Weiss (1986) households face explicit borrowing limits and cannot insure themselves against idiosyncratic shocks. Agents’ desire to self-insure through precautionary savings creates a demand for the single asset, money. In our model households can hold money and physical capital. The return on capital is risky and its risk profile differs from the endogenous risk profile of money. Financial institutions create inside money and mitigate financial frictions. In Kiyotaki and Moore (2008) money and capital coexist. Money is desirable as it does not suffer from a resellability constraint as physical capital does. Lippi and Trachter (2012) characterize the trade-off between insurance and production incentives of liquidity provision. Levin (1991) shows that monetary policy is more effective than fiscal policy if the government does not know which agents are productive. The finance papers by Diamond and Rajan (2006) and Stein (2012) also address the role of

\[1\] The literature on credit channels distinguishes between the bank lending channel and the balance sheet channel (financial accelerator), depending on whether banks or corporates/households are capital constrained. Strictly speaking our setting refers to the former, but we are agnostic about it and prefer the broader credit channel interpretation.
monetary policy as a tool to achieve financial stability.

More generally, there is a large macro literature which also investigated how macro shocks that affect the balance sheets of intermediaries or end-borrowers become amplified and affect the amount of lending and the real economy. These papers include Bernanke and Gertler (1989), Kiyotaki and Moore (1997) and Bernanke, Gertler and Gilchrist (1999), who study financial frictions using a log-linearized model near steady state. In these models shocks to intermediary/end-borrower net worths affect the efficiency of capital allocation and asset prices. However, log-linearized solutions preclude volatility effects and lead to stable system dynamics. Brunnermeier and Sannikov (2014) study the full equilibrium (risk) dynamics, focusing on the differences in system behavior near the steady state, and away from it. They find that the system is stable to small shocks near the steady state, but large shocks make the system unstable and generate systemic endogenous risk. Thus, system dynamics are highly nonlinear. Large shocks have much more serious effects on the real economy than small shocks. He and Krishnamurthy (2013) also study the full equilibrium dynamics and focus in particular on credit spreads. In Mendoza and Smith’s (2006) international setting the initial shock is also amplified through a Fisher debt-disinflation that arises from the interaction between domestic agents and foreign traders in the equity market. In our paper debt disinflation is due to the appreciation of inside money. For a more detailed review of the literature we refer to Brunnermeier et al. (2013).

This paper is organized as follows. Section 2 sets up the model and derives first the solutions for two polar cases. Section 3 presents computed examples and discusses equilibrium properties, including capital and money value dynamics, the amount of lending through intermediaries, and the money multiplier for various parameter values. Section 4 introduces long-term bonds and studies the effect of interest-rate policies as well as open-market operations. Section 5 showcases a numerical example of monetary policy. Section 6 concludes.

2 The Baseline Model Absent Policy Intervention

The economy is populated by two types of agents: households and intermediaries. Each household can use capital to produce either good $a$ or good $b$, but can only manage a single project at a time. Each project carries both idiosyncratic and aggregate good-specific risk. The two goods are then combined into an aggregate good that can be consumed or invested. Intermediaries help fund households that produce good $b$ by buying their equity. Intermediaries pool these equity stakes in order to diversify the idiosyncratic risk, and obtain
funding for these holdings by accepting money deposits. Households that produce good \( a \) cannot issue equity to intermediaries.

Households can split their wealth between one project of their choice and money. There is outside money - currency, whose nominal supply is fixed in the absence of monetary policy - and inside money - currency claims issued by intermediaries to finance their investments in equity of households that use technology \( b \). However, while the nominal supply of outside money is fixed, the real value of money is determined endogenously in equilibrium. The dynamic evolution of the economy is driven by the effect of shocks on the agents’ wealth distribution, as reflected through their portfolio choice. The model is solved using standard portfolio choice theory, except that asset prices - including the price of money - are endogenous.

**Technologies.** All physical capital \( K_t \) in the world is allocated between the two technologies. If capital share \( \psi_t \in [0, 1] \) is devoted to produce good \( a \), then goods \( a \) and \( b \) combined make \( A(\psi)K_t \) of the aggregate good. Function \( A(\psi) \) is concave and has an interior maximum, an example is the standard technology with constant elasticity of substitution \( s \),

\[
A(\psi) = A \left( \frac{1}{2} \psi \left( \frac{\psi}{s} \right)^{\frac{1}{s-1}} + \frac{1}{2} (1 - \psi) \left( \frac{1}{s} \right)^{\frac{1}{s-1}} \right)^{\frac{1}{s-1}}.
\]

In competitive markets, prices of goods \( a \) and \( b \) reflect their marginal contributions to the aggregate good. Prices must be such that a unit of capital employed in each sector produces output valued at

\[
A^a(\psi) = (1 - \psi) A'(\psi) + A(\psi) \quad \text{and} \quad A^b(\psi) = -\psi A'(\psi) + A(\psi),
\]

respectively.\(^3\)

\(^2\)For \( s = \infty \) the outputs are perfect substitutes, for \( s = 0 \) there is no substitutability at all, while for \( s = 1 \) the substitutability corresponds to that of a Cobb-Douglas production function.

\(^3\)If total output is \( A(\psi)K \), then an \( \epsilon \) amount of capital devoted to technology \( a \) would change total productivity to

\[
A \left( \frac{\psi K + \epsilon}{K + \epsilon} \right) (K + \epsilon).
\]

Differentiating with respect to \( \epsilon \) at \( \epsilon = 0 \), we obtain

\[
A'(\psi) \frac{K + \epsilon - (\psi K + \epsilon)}{(K + \epsilon)^2} (K + \epsilon) + A(\psi) = A'(\psi)(1 - \psi) + A(\psi).
\]

Likewise, the marginal contribution of capital devoted to technology \( b \) would be \( A(\psi) - \psi A'(\psi) \). The weighted sum of the two terms is \( A(\psi) \) since the production technology is homogenous of degree 1.
Physical capital $k_t$ is subject to shocks that depend on the technology in which it is employed. If used in technology $a$ capital follows

$$
\frac{d k_t}{k_t} = (\Phi(\iota_t) - \delta) \, dt + \sigma^a \, d Z^a_t + \tilde{\sigma}^a \, d \tilde{Z}_t,
$$

(2.1)

where $d Z^a_t$ are the sector-wide Brownian shocks and $d \tilde{Z}_t$ are project-specific shocks, independent across agents, which cancel out in the aggregate. A similar equation applies if capital is used in technology $b$. Sector-wide shocks $d Z^a_t$ and $d Z^b_t$ are independent of each other. The investment function $\Phi$ has the standard properties $\Phi' > 0$ and $\Phi'' \leq 0$, and the input for investment $\iota_t$ is the aggregate good.

**Preferences.** All agents have identical logarithmic preferences with a common discount rate $\rho$. That is, any agent maximizes the expected utility of

$$
E \left[ \int_0^\infty e^{-\rho t} \log c_t \, dt \right],
$$

subject to individual budget constraints, where $c_t$ is the consumption of the aggregate good at time $t$.

**Financing Constraints.** Households can hold money and invest in either technology $a$ or technology $b$. They can issue risky claims only towards the intermediary sector (not to each other). However, the amount of risk they can offload to the intermediary sector is bounded above, with bounds $\chi^a$ and $\chi^b$ satisfying $0 \leq \chi^a < \chi^b \leq 1$. For simplicity, we set in our baseline model $\chi^a = 0$, and then denote $\chi \equiv \chi^b$, with $\chi$ near 1. Intermediaries finance their risky holdings (households’ outside equity) by issuing claims (nominal IOUs) with return identical to the return on money. These claims, or inside money, are as safe as currency, outside money. In the baseline model, there is a fixed amount of outside fiat money in the economy that pays zero interest. Figure 1 provides a schematic representation of the basic financing structure of the model.\(^4\)

\(^4\)Notice that if $\chi^a = \chi^b$, then by holding this maximum fraction of equity of each sector, intermediaries guarantee that the fundamental risk of their assets is proportional to the risk of the economy as a whole. In this case, intermediaries end up perfectly hedged, as the risk of money is also proportional to the risk of the whole economy and the intermediaries’ wealth share follows a deterministic path. In contrast, if $\chi^a < \chi^b$, then intermediaries are always overexposed to the risk of sector $b$. In this case, they hold the maximum amount $\chi^a$ of equity of sector $a$, as this helps them hedge and also helps households in sector $a$ offload aggregate risk. They also hold more than fraction $\chi^a$ of equity of sector $b$, as the risk premium they demand is initially second-order, and households in sector $b$ demand insurance.

\(^5\)The model could be easily enriched to allow intermediaries to sell off part of the equity claims up to a
Finally let us offer some additional brief remarks on model interpretation. First, since outside money and inside money have the same return and risk profile, it is equivalent whether households hold outside money or the intermediary/financial sector holds outside money and issues a corresponding amount of inside money. Second, we interpret our intermediary/financial sector as a sector that includes traditional banking, but also shadow banking and other forms of intermediation and risk mitigation. And third, as all households have some money balances, the model has no clear borrowing or lending sectors. This feature distinguishes our model from more conventional loanable funds models.

**Assets, Returns and Portfolios.** Let $q_t$ denote the price of physical capital per unit relative to the numeraire, the aggregate consumption good. In this paper we do not consider equilibria with jumps, so let us postulate for now that $q_t$ follows a Brownian process of the form

$$\frac{dq_t}{q_t} = \mu_t^q \, dt + (\sigma_t^q)^T \, dZ_t,$$

where $dZ_t = [dZ_t^a, dZ_t^b]^T$ is the vector of aggregate shocks. Then the capital gains component of the return on capital, $d(k_t q_t)/(k_t q_t)$, can be found using Ito’s lemma. The dividend yield is $(A^a(\psi) - \iota_t)/q_t$ for technology $a$ and $(A^b(\psi) - \iota_t)/q_t$ for technology $b$.

$\chi^I < 1$. This would not alter the qualitative results of the model.
The total (real) return of an individual project in technology $a$ is

$$dr_t^a = \frac{A^a(\psi_t) - \iota_t}{q_t} dt + (\Phi(\iota_t) - \delta + \mu^a_t + (\sigma^a_t)^T \sigma^a 1^a) dt + (\sigma^a_t + \sigma^a 1^a)^T dZ_t + \bar{\sigma}^a d\tilde{Z}_t,$$

where $1^a$ is the column coordinate vector with a single 1 in position $a$. The (real) return in technology $b$ is written analogously. The optimal investment rate $\iota_t$, which maximizes the return of any technology, is given by the first-order condition $1/q_t = \Phi'(\iota_t)$. We denote the investment rate that satisfies this condition by $\iota(q_t)$.

The return on technology $b$ is split between households who hold inside equity and earn $dr_t^{bH}$ and intermediaries who hold outside equity and earn $dr_t^{bI}$, so

$$dr_t^b = (1 - \chi_t) dr_t^{bH} + \chi_t dr_t^{bI},$$

where $\chi_t \leq \bar{\chi}$ is the fraction of outside equity households issue. The two types of equity have identical risks, but potentially different returns. The required return on inside equity may be higher if households would like to issue more outside equity but cannot due to the constraint. That is, in equilibrium we have we have $dr_t^{bH} \geq dr_t^{bI}$, with equality if $\chi_t < \bar{\chi}$.

Total money supply is fixed absent monetary policy. The value of all money depends on the size of the economy. Denote the real value of all outside money by $p_t K_t$. Since inside money is a liability for the intermediary sector and an asset for the household sector, it nets out overall. Let us postulate that $p_t$ follows a Brownian process of the form

$$\frac{dp_t}{p_t} = \mu^p_t dt + (\sigma^p_t)^T dZ_t.$$  \hfill (2.3)

The law of motion of aggregate capital is

$$\frac{dK_t}{K_t} = (\Phi(\iota_t) - \delta) dt + \psi_t \sigma^a dZ^a_t + (1 - \psi_t) \sigma^b dZ^b_t,$$  \hfill (2.4)

and the return on money, the real interest rate, is given just by the capital gains rate

$$dr_t^M = \frac{d(p_t K_t)}{p_t K_t} = (\Phi(\iota_t) - \delta + \mu^p_t + (\sigma^p_t)^T \sigma^K_t) dt + (\sigma^K_t + \sigma^p_t)^T dZ_t.$$
When a household chooses to produce good \( a \), its net worth follows

\[
\frac{dn_t}{n_t} = x_t^a dr_t^a + (1 - x_t^a) dr_t^M - \zeta_t^a dt, \tag{2.5}
\]

where \( x_t^a \) is the portfolio weight on capital and \( \zeta_t^a \) is its propensity to consume (i.e. consumption per unit of net worth).

The net worth of a household who produces good \( b \) follows

\[
\frac{dn_t}{n_t} = x_t^b dr_t^{bH} + (1 - x_t^b) dr_t^M - \zeta_t^b dt. \tag{2.6}
\]

Households can choose whether to work in sector \( a \) or \( b \), that is, in equilibrium they must be indifferent with respect to this choice. Denote by \( \alpha_t \) the net worth of households who specialize in sector \( a \), as a fraction of total household net worth.

The net worth of an intermediary follows

\[
\frac{dn_t}{n_t} = x_t \bar{r}_t^{bI} + (1 - x_t) dr_t^M - \zeta_t dt, \tag{2.7}
\]

where \( \bar{r}_t^{bI} \) denotes the return on households’ outside equity \( dr_t^{bI} \) with idiosyncratic risk diversified away, i.e. removed. If intermediaries use leverage, i.e. issue inside money, then of course \( x_t > 1 \).

**Equilibrium Definition.** The agents start initially with some endowments of capital and money. Over time, they trade - they choose how to allocate their wealth between the assets available to them. That is, they solve their individual optimal consumption and portfolio choice problems to maximize utility, subject to the budget constraints (2.5), (2.6) and (2.7). Individual agents take prices as given. Given prices, markets for capital, money and consumption goods have to clear.

If the net worth of intermediaries is \( N_t \), then given the world wealth of \( (q_t + p_t)K_t \), the intermediaries’ net worth share is denoted by

\[
\eta_t = \frac{N_t}{(q_t + p_t)K_t}. \tag{2.8}
\]

**Definition.** Given any initial allocation of capital and money among the agents, an equilibrium is a map from histories \( \{Z_s, s \in [0, t]\} \) to prices \( p_t \) and \( q_t \), return differential \( dr_t^{bH} = dr_t^{bI} \geq 0 \), the households’ wealth allocation \( \alpha_t \), equity allocation \( \chi_t \leq \bar{\chi} \), portfolio
weights \((x^a_t, x^b_t, x)\) and consumption propensities \((\zeta^a_t, \zeta^b_t, \zeta_t)\), such that

(i) all markets, for capital, equity, money and consumption goods, clear,

(ii) all agents choose technologies, portfolios and consumption rates to maximize utility

(households who produce good \(b\) also choose \(\chi_t\)).

One important choice here is that of households: each household can run only one project either in technology \(a\) or \(b\). They must be indifferent between the two choices. Households who choose to produce good \(b\) must also choose how much equity to issue. If outside equity earns less than the return of technology \(b\), these households would want to issue the maximal amount of outside equity of \(\chi_t = \bar{\chi}\). This happens in equilibrium only if intermediaries are willing to accept this supply of equity at a return discount, i.e. \(dr^b_H < dr^b_I\), so that inside equity earns a premium. This is the case only if the intermediaries are well-capitalized. Otherwise, \(dr^b_H = dr^b_I = dr^b\), i.e. inside and outside equity of technology \(b\) earns the same return as technology \(b\). In this case, households are indifferent with respect to the amount of equity they issue, so the equity issuance constraint does not bind.

\[ \text{2.1 Equilibrium Conditions} \]

Logarithmic utility has two well-known tractability properties. First, an agent with logarithmic utility and discount rate \(\rho\) consumes at the rate given by \(\rho\) times net worth. Thus, \(\zeta_t = \zeta^a_t = \zeta^b_t = \rho\) and the market-clearing condition for consumption goods is

\[ \rho(q_t + p_t)K_t = (A(\psi_t) - \tau_t)K_t. \] (2.9)

Second, the excess return of any risky asset over any other risky asset is explained by the covariance between the difference in returns and the agent’s wealth.

From (2.5) and (2.6), the wealth of households in sectors \(a\) and \(b\) is exposed to aggregate risk of

\[ \sigma^N_a = x^a_t \left( \sigma^a 1^a + \sigma^q - \sigma^M \right) + \sigma^M \] and \[ \sigma^N_b = x^b_t \left( \sigma^b 1^b + \sigma^q - \sigma^M \right) + \sigma^M, \]

and idiosyncratic risk of \(x^a_t \tilde{\sigma}^a\) and \(x^b_t \tilde{\sigma}^b\), respectively. Consequently, the difference between
expected returns of technology $a$ and money is given by

\[
\frac{E_t[dr^a_t - dr^M_t]}{dt} = (\nu^a_t)^T \sigma^N_t + x^a_t (\tilde{\sigma}^a)^2,
\]

where the right-hand side is the covariance of the net worth of a household in sector $a$ with the excess risk of technology $a$ over money.

To write an analogous condition for technology $b$, we have to take into account the split of risk between households and intermediaries. Note that the net worth of intermediaries is exposed to risk

\[
\sigma^N_t = x_t \nu^b_t + \sigma^M_t.
\]

Therefore, the expected excess return of technology $b$ must satisfy

\[
\frac{E_t[dr^b_t - dr^M_t]}{dt} = (1 - \chi_t)((\nu^b_t)^T \sigma^N_t + x^b_t (\tilde{\sigma}^b)^2) + \chi_t (\nu^b_t)^T \sigma^N_t
\]

The difference in return of inside and outside equity of households in sector $b$ is then

\[
\frac{dr^b_{tH} - dr^b_{tI}}{dt} = (\nu^b_t)^T \sigma^N_t + x^b_t (\tilde{\sigma}^b)^2 - (\nu^b_t)^T \sigma^N_t \geq 0,
\]

with equality if $\chi_t < \bar{\chi}$.

Households must be indifferent between investing in technologies $a$ and $b$. The following proposition summarizes the relevant condition

**Proposition 1.** In equilibrium

\[
(x^a_t)^2 (|\nu^a_t|^2 + (\tilde{\sigma}^a)^2) = (x^b_t)^2 (|\nu^b_t|^2 + (\tilde{\sigma}^b)^2).
\]

**Proof.** See Appendix. \qed

Market clearing for capital implies that portfolio weights, given the net worth shares of intermediaries and households, have to be consistent with the allocation of the fraction $\psi_t$ of capital to technology $b$. Denote by

\[
\varphi_t = \frac{p_t}{q_t + p_t}
\]
The fraction of the world wealth that is in the form of money. Then

\[ x_t = \frac{\chi_t \psi_t (1 - \vartheta_t)}{\eta_t}. \]  

(2.13)

Furthermore, the net worth of households who employ technologies \( a \) and \( b \), together, must add up to \( 1 - \eta_t \), i.e.,

\[ \frac{(1 - \psi_t)(1 - \vartheta_t)}{x_a^t} + \frac{\psi_t (1 - \chi_t)(1 - \vartheta_t)}{x_b^t} = 1 - \eta_t, \]  

(2.14)

and the fraction household wealth in sector \( a \) is given by

\[ \alpha_t = \frac{(1 - \psi_t)(1 - \vartheta_t)}{x_a^t (1 - \eta_t)}. \]

**2.2 Evolution of the State Variable**

Finally, we have to describe how the state variable \( \eta_t \), which determines prices of capital and money \( p_t \) and \( q_t \), evolves over time. The law of motion of \( \eta_t \), together with the specification of prices and allocations as functions of \( \eta_t \), constitute the full description of equilibrium: i.e. the map from any initial allocation and a history of shocks \( \{ Z_s s \in [0, t] \} \) into the description of the economy at time \( t \) after that history. The following proposition characterizes the equilibrium law of motion of \( \eta_t \).

**Proposition 2.** The equilibrium law of motion of \( \eta_t \) is given by

\[ \frac{d \eta_t}{\eta_t} = (1 - \eta_t) \left( x_t^2 |\nu_t^b|^2 - (x_t^a)^2 (|\nu_t^a|^2 + (\tilde{\sigma}^a)^2) \right) dt + (x_t \nu_t^b + \sigma_t^\theta)^T (\sigma_t^\theta dt + dZ_t). \]  

(2.15)

The law of motion of \( \eta_t \) is so simple because the earnings of intermediaries and households can be expressed in terms of risks they take and the required equilibrium risk premia. The first term on the right-hand side reflects the relative earnings of intermediaries and households. The second term on the right-hand side of (2.15) reflects mainly the volatility of \( \eta_t \), due to the imperfect risk sharing between intermediaries and households.

**Proof.** The law of motion of total net worth of intermediaries, given the risks that they take,
must be
\[
\frac{dN_t}{N_t} = dr_t^M - \rho \, dt + x_t(\nu_t^h)^T (x_t(\nu_t^h + \sigma_t^M) \, dt + dZ_t).
\]  \hspace{1cm} (2.16)

The law of motion of world wealth \((q_t + p_t)K_t\), the denominator of (2.8), can be found from the total return on world wealth, after subtracting the dividend yield of \(\rho\) (i.e., aggregate consumption). To find the returns, we take into account the risk premia that various agents earn. We have
\[
\frac{d((q_t + p_t)K_t)}{(q_t + p_t)K_t} = dr_t^M - \rho \, dt + (1 - \vartheta_t) (\sigma_t^K + \sigma_t^\rho - \sigma_t^M)^T \, dZ_t +
\]
\[
(1 - \vartheta_t) ((1 - \psi_t)(\nu_t^a)^T \sigma_t^N + x_t^a(\sigma_t^\nu^a)^2) + \psi_t (\chi_t(\nu_t^b)^T \sigma_t^N + (1 - \chi_t)(\nu_t^b)^T \sigma_t^N + x_t^b(\sigma_t^\nu^b)^2)) \, dt.
\]

Recall that
\[
\sigma_t^N = x_t^b + \sigma_t^M, \quad \sigma_t^N = x_t^a + \sigma_t^M \quad \text{and} \quad \sigma_t^N = x_t^b + \sigma_t^M
\]

and note that
\[
(1 - \psi_t)\nu_t^a + \psi_t\nu_t^b = \sigma_t^q - \sigma_t^\rho.
\]

Therefore, the law of motion of aggregate wealth can be written as
\[
\frac{d((q_t + p_t)K_t)}{(q_t + p_t)K_t} = dr_t^M - \rho \, dt + (1 - \vartheta_t) (\sigma_t^q - \sigma_t^\rho)^T (\sigma_t^M \, dt + dZ_t) +
\]
\[
(1 - \vartheta_t) ((1 - \psi_t)x_t^a(\nu_t^a)^2 + (\sigma_t^\nu^a)^2) + \psi_t (\chi_t x_t^b(\nu_t^b)^2 + (1 - \chi_t)x_t^b(\sigma_t^\nu^b)^2)) \, dt =
\]
\[
\frac{dN_t}{N_t} = dr_t^M - \rho \, dt - (\sigma_t^q)^T (\sigma_t^M \, dt + dZ_t) + \eta_t x_t^2|x_t^b|^2 \, dt + (1 - \eta_t)(x_t^a)^2(\sigma_t^\nu^a)^2 \, dt, \hspace{1cm} (2.17)
\]
where we used (2.14) and the indifference condition of Proposition 1.

Thus, using Ito’s lemma, we obtain (2.15).\footnote{Ito’s lemma implies that \(\sigma_t^p = (1 - \vartheta)(\sigma_t^p - \sigma_t^\rho)\) and \(\mu_t^p = (1 - \vartheta)(\mu_t^p - \mu_t^\rho) - \sigma_t^p \sigma_t^\rho + (\sigma_t^\rho)^2.\)}

\footnote{If processes \(X_t\) and \(Y_t\) follow
\[
dX_t/X_t = \mu_t^X \, dt + \sigma_t^X \, dZ_t \quad \text{and} \quad dY_t/Y_t = \mu_t^Y \, dt + \sigma_t^Y \, dZ_t,
\]}

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3 Model without Intermediaries

The goal of this section is to understand the determinants of the value of money in a model without intermediaries. The key determinant of the value of money is, of course, the level of idiosyncratic risk.

We can anticipate properties of full equilibrium dynamics through our understanding of the economy without intermediaries. Since intermediaries reduce the amount of idiosyncratic risk in the economy, the presence of a healthy intermediary sector is akin to a reduction in idiosyncratic risk parameters in the model without intermediaries.

3.1 Value and Risk of Money

Assume that $\eta = 0$. Suppose for the sake of simplicity that $\sigma^a = \sigma^b = \sigma$, $\tilde{\sigma}^a = \tilde{\sigma}^b = \tilde{\sigma}$ and that $\max_{\psi} A(\psi) = \bar{A}$ is maximized at $\psi = 1/2$. Then half of all households produce good $a$, and the rest, good $b$. Aggregate capital in the economy follows

$$\frac{dK_t}{K_t} = (\Phi(\iota_t) - \delta) \, dt + \frac{\sigma}{2} dZ_t^a + \frac{\sigma}{2} dZ_t^b.$$

Prices $p$ and $q$ are constant. The volatility of the money (or the whole economy) and the incremental risk of a project in either sector (orthogonal to the risk of money) are

$$\bar{\sigma} \equiv \sqrt{\sigma^2/2} \quad \text{and} \quad \hat{\sigma} \equiv \sqrt{\tilde{\sigma}^2 + \sigma^2/2},$$

respectively. Note that the total risk of technology $a$ or $b$ is $\sqrt{\bar{\sigma}^2 + \hat{\sigma}^2} = \sqrt{\sigma^2 + \tilde{\sigma}^2}$.

Effectively, the economy is equivalent to a single-good economy with aggregate risk $\bar{\sigma}$ and project-specific risk $\hat{\sigma}$. In this economy, the market-clearing condition for output (2.9) becomes

$$\bar{A} - \iota(q) = \rho \left( \frac{p + q}{q/(1-\vartheta)} \right). \quad (3.1)$$

Each household puts portfolio weight $1-\vartheta$ on capital, so its net worth is exposed to aggregate risk $\bar{\sigma}$ and project-specific risk $(1-\vartheta)\hat{\sigma}$. The excess return on capital over money is the dividend yield $(\bar{A} - \iota(q))/q$, since the capital gains rates are the same. Therefore, the asset-
Pricing condition of capital relative to money is

\[
\frac{\bar{A} - \iota(q)}{q} = (1 - \vartheta)\hat{\sigma}^2 \quad \Rightarrow \quad \vartheta = 1 - \sqrt{\rho}/\hat{\sigma}.
\] (3.2)

Equilibrium in which money has positive value exists only if \(\hat{\sigma}^2 > \rho\). As \(\hat{\sigma}\) increases, the value of money relative to capital rises.

For a special form of the investment function \(\Phi(\iota) = \log(\kappa\iota + 1)/\kappa\), we can also get closed-form expressions for the equilibrium prices of money and capital.\(^8\) Then (3.1) implies that

\[
q = \frac{\kappa\bar{A} + 1}{\kappa\sqrt{\rho}\hat{\sigma} + 1} \quad \text{and} \quad p = \frac{\hat{\sigma} - \sqrt{\rho}}{\sqrt{\rho}}q.
\] (3.3)

There is always an equilibrium in which money has no value. In that equilibrium the price of capital satisfies \(\bar{A} - \iota(q) = \rho q\), so that

\[
q = \frac{\kappa\bar{A} + 1}{\kappa\rho + 1}.
\] (3.4)

Then the dividend yield on capital is \((\bar{A} - \iota_t)/q = \rho\) and expected return on capital is \(\rho + \Phi(\iota_t) - \delta\). Subtracting the idiosyncratic risk premium of \(\hat{\sigma}^2\) the required return on an asset that carries the same risk as the whole economy, or \(K_t\), is

\[
\rho - \hat{\sigma}^2 + \Phi(\iota_t) - \delta.
\]

If this rate is lower than the growth rate of the economy, i.e. \(\Phi(\iota_t) - \delta\), then an equilibrium in which money has positive value exists. Lemma 1 in the Appendix generalizes these results to the case when \(\sigma^a \neq \sigma^b\) and \(\hat{\sigma}^a \neq \hat{\sigma}^b\).

These closed-form solutions allow us to anticipate how the value of money may fluctuate in an economy with intermediaries. When \(\eta_t\) approaches 0, households face high idiosyncratic risk in both sectors, leading to a high value of money. In contrast, when \(\eta_t\) is large enough, then most of idiosyncratic risk is concentrated in sector \(a\), as households in sector \(b\) pass on the idiosyncratic risk to intermediaries. This leads to a lower value of money.

Intermediary net worth and the value of money will generally fluctuate due to aggregate shocks \(Z^a\) and \(Z^b\). Relative to world wealth - recall that \(\eta_t\) measures the intermediary net

---

\(^8\) When the investment adjustment cost parameter \(\kappa\) is close to 0, i.e. \(\Phi(\iota)\) is close to 1, then the price of capital \(q\) goes to 1 (this is Tobin’s \(q\)). As \(\kappa\) becomes large, the price of capital depends on dividend yield \(\bar{A}\) relative to the discount rate \(\rho\) and the level of idiosyncratic risk that affects the value of money.
worth relative to total wealth - intermediaries are long shocks $Z^b$ and short shocks $Z^a$ when they invest in equity of households who produce good $b$. A fundamental assumption of our model is that intermediaries cannot hedge this aggregate risk exposure. Due to this, they may suffer losses, and losses force them to stop investing in equity of households who use technology $b$. The intermediary sector may become undercapitalized.

**Impossibility of “As If” Representative Agent Economies.** Note that it is impossible to construct an “as if” representative agent economy with the same aggregate output and investment streams and same asset prices that mimics the equilibrium outcome of our heterogeneous agents economy. In any representative agent economy, absence of individual-level idiosyncratic risk, capital returns strictly dominate money and hence money could never have some positive value.

### 3.2 Welfare Analysis

We start with a general result, which allows us to compute welfare of agents with logarithmic utility. Expression (3.6) below is valid for an arbitrary process (3.5), regardless of whether it arises from a feasible equilibrium trading strategy or not.\(^9\)

**Proposition 3.** Consider an agent who consumes at rate $\rho n_t$ where $n_t$ follows

$$\frac{dn_t}{n_t} = \mu^n_t \, dt + \sigma^n_t \, dZ_t \tag{3.5}$$

Then the agent’s expected future utility at time $t$ takes the form

$$E_t \left[ \int_t^\infty e^{-\rho(s-t)} \log(\rho n_s) \, ds \right] = \log(\rho n_t) \rho + \frac{1}{\rho} E_t \left[ \int_t^\infty e^{-\rho(s-t)} \left( \mu^n_s - \frac{|\sigma^n_s|^2}{2} \right) \, ds \right]. \tag{3.6}$$

**Proof.** See Appendix. \(\square\)

Without intermediaries, drift and volatility of wealth for all households are time-invariant. In general, given portfolio weights $1 - \vartheta$ on capital and $\vartheta$ on money, we have

$$\mu^n = (1 - \vartheta) \frac{\bar{A} - \iota(q)}{q} + \Phi(\iota(q)) - \delta - \rho, \quad \sigma^n = \sqrt{(1 - \vartheta)^2 \tilde{\sigma}^2 + \bar{\sigma}^2}. \tag{3.7}$$

\(^9\)For example, we can use (3.5) to evaluate welfare of a hypothetical representative agent, who consumes a portion of world output, to estimate welfare that could be attained without idiosyncratic risk.
For the equilibrium value of $\vartheta$ given by (3.2), we have

$$
\mu^n = \Phi(\iota(q)) - \delta \quad \text{and} \quad \sigma^n = \sqrt{\rho + \bar{\sigma}^2}.
$$

(3.8)

Combining (3) with (3.8), we get the following proposition

**Proposition 4.** Suppose $\hat{\sigma}^2 > \rho$, so that monetary equilibrium exists in the economy without intermediaries. Then in this equilibrium, the welfare of a household with initial wealth $n_0 = 1$ is

$$
U^H = \frac{\log(\rho)}{\rho} + \frac{\Phi(\iota(q)) - \delta - (\rho + \bar{\sigma}^2)/2}{\rho^2}.
$$

Macro-prudential regulation. How does welfare in equilibrium with money compare to welfare in the money-less equilibrium? If the regulator can control the value of money by specifying a money holding requirement of the agents, will the money under optimal policy have greater value than in equilibrium, or lower value? Note that higher value of money allows agents to reduce their idiosyncratic risk exposure, but creates a distortion on the investment front, since the value of capital becomes lower.

What if the regulator can control $\vartheta$ by forcing the agents to hold specific amounts of money? As it turns out, under some mild restrictions on $\vartheta$, it will be optimal for the planner to force agents to hold more money. Our results are summarized in the following proposition.

**Proposition 5.** Assume that $\Phi(\iota) = \log(\kappa \iota + 1)/\kappa$. Then if money can have positive equilibrium value, welfare in equilibrium with money is always greater than that in the moneyless equilibrium. Furthermore, relative to the value of $\vartheta$ in the equilibrium with money, optimal policy raises $\vartheta$ if and only if

$$
\hat{\sigma}(1 - \kappa \rho) < 2 \sqrt{\rho}.
$$

(3.9)

**Proof.** See Appendix. \qed

Condition (3.9) reflects the trade-off between the role of money as an insurance asset, and the distortionary effect of rising money value on investment. On the one hand, the returns to money are free of idiosyncratic risk, so individual households have less exposure to their own individual-specific shocks, improving welfare. On the other hand, in the money equilibrium, the price of capital is lower, so investment is lower, so overall growth is lower. When adjustment costs $\kappa$ are large enough, these distortions are minimal, so the diversification benefit dominates, as we see in condition (3.9).
4 Analysis with Intermediaries

In this section, we analyze the full model economy with intermediaries. Intermediaries are diversifiers, allowing households that invest in technology $b$ to offload some of their idiosyncratic risk. The capacity of intermediaries to act as “diversifiers” depends on their capitalization, and so it is not surprising that aggregate economic activity also depends on intermediary capitalization. Since intermediaries are exposed (in a levered way) to the idiosyncratic risk of technology $b$, their wealth share moves over time, as different $a$-shocks and different $b$-shocks hit the economy.

In the previous section, we considered the extreme polar case when intermediary capitalization is 0. In that case, in the money equilibrium, the value of money is high – it is an attractive insurance vehicle for households invested in either of the two technologies. In contrast, with a functioning intermediary sector, households that invest in technology $b$ can offload some of their idiosyncratic risk, so there is less demand for insurance vehicles. As a result money is less attractive and so its real value is low. At the other end of the spectrum, $\eta_t$ can, however, also be too high: When $\eta_t$ is close to 1 there is too much focus on the sector $b$ good and so aggregate economic activity declines.

The rest of this section proceeds as follows. First, we provide a full characterization of the equilibrium of our economy. Second, we conduct welfare analysis.

4.1 Equilibrium

The computational procedure we employ, both with and without monetary policy, is described in Appendix .... Consider parameter values $\rho = 0.05$, $A = 0.5$, $\sigma^a = \sigma^b = 0.4$, $\tilde{\sigma}^a = 0.6$, $\tilde{\sigma}^b = 1.2$, $s = 0.8$, $\Phi(\iota) = \log(\kappa \iota + 1) / \kappa$ with $\kappa = 2$, and $\bar{\chi} = 0.999$.

We start by looking at the allocation of capital. The production of good $b$ depends on intermediaries, it increases in the net worth share of the intermediary sector $\eta$. When $\eta$ drops, the risk premia that intermediaries demand for equity stakes in projects of households in sector $b$ rise, to the point that the households may be willing to sell less than fraction $\bar{\chi}$ of outside equity. See Figure 2.

Figure 3 shows the prices $p(\eta)$ and $q(\eta)$ of money and capital in equilibrium. At $\eta = 0$, the values of $p$ and $q$ converge to those under the benchmark without intermediaries, $q = 1.0532$ and $p = 3.4151$. As $\eta$ rises, the price of capital rises and the price of money drops
(although the price of capital drops again near $\eta = 1$). Money becomes less valuable as $\eta$ rises mainly because intermediaries create money. The inside money on the liabilities sides of the intermediaries’ balance sheets is a perfect substitute to outside money.

**Volatility of $\eta$.** Figure 4 illustrates the equilibrium dynamics through the drift and volatility of the state variable $\eta$. From Proposition 2, the volatility of $\eta_t$ is given by

$$\sigma^\eta \eta = \eta \left( x_t (\sigma^b 1^b - \sigma^K_t) + \sigma_t^\vartheta \left( 1 - \frac{x_t}{1 - \vartheta_t} \right) \right) \tag{4.1}$$

Variable $\eta_t$ has volatility for two reasons: from the mismatch between the fundamental risk of assets that intermediaries hold, $\sigma^b dZ^b_t$, and the overall fundamental risk in the economy $\sigma^K_t dZ_t$, and from amplification because of the endogenous fluctuations of $\vartheta(\eta_t)$ (the price of money relative to capital). As long as the intermediaries’ portfolio share of households’ equity is greater than $1 - \vartheta_t$, the world capital share, and as long as $\vartheta'(\eta) < 0$, amplification exists. Figure 4 shows both the portion of the volatility of $\eta_t$ that arises from fundamental risk only, and total volatility that includes the effects of amplification. Amplification becomes prominent when intermediaries are undercapitalized.
Liquidity and Disinflationary Spiral. Intermediaries’ wealth share volatility is amplified by two adverse feedback loops/spirals when \( \eta \) is low. First, the traditional amplification channel works on the asset sides of the intermediary balance sheets: as the price of physical capital \( q(\eta) \) drops following a negative shock. Second, shocks hurt intermediaries on the liability side of the balance sheets through the Fisher disinflationary spiral. The real value of intermediaries’ liabilities rises. Both effects impair intermediaries’ net worth. Intermediaries’ response to these losses is to shrink their balance sheet further, which leads to yet more fire-sales (lowering the price \( q \)) and reduction in inside money (increasing the value of liabilities \( p \)). In other words, they take fewer deposits, create less inside money, and the money multiplier collapses.\(^1\) This again reduces their net worth, and so on.

By rewriting equation (4.1), we can find a mathematical representation of these spirals in our model.

\[
\sigma^\eta_t = \frac{x_t(\sigma^b 1^b - \sigma^K_t)}{1 - \left( \frac{\psi^\prime \chi_t - \eta_t}{\eta_t} \right) \frac{-\psi'(\eta_t)}{\psi/\eta_t}}
\]

The numerator reflects the amount of fundamental risk the intermediary sector is exposed

\(^1\)In reality, rather than turning savers away, financial intermediaries might still issue demand deposits and simply park the proceeds with the central bank as excess reserves.
to. The denominator \((1 - ...\) comes from the sum of a geometric series, a mathematical manifestation of the spiral. The term \(\frac{\psi_t - \chi_t}{\eta_t}\) corresponds to the intermediary sector’s leverage ratio, while \(\frac{\psi_t}{\eta_t}\) is the elasticity of \(\psi(\cdot)\) w.r.t. the wealth share \(\eta\). The formula clearly shows that amplification increases with intermediaries’ leverage and with the elasticity (in a multiplicative fashion). The function \(\psi(\eta) = p(\eta)/(q(\eta) + p(\eta))\) subsumes these two spirals. The liquidity spiral works through \(q(\eta)\), while the disinflationary spiral works through \(p(\eta)\). Figure 3 shows that both spirals are adverse: a negative shock that lowers \(\eta\) lowers the value of assets \(q(\eta)\), the liquidity spiral, and increases the value of liabilities \(p(\eta)\), the disinflationary spiral.

\[
\mu^\eta_t = \eta(1 - \eta)(x_t^2|\nu_t^b|^2 - (x_t^a)^2(|\nu_t^a|^2 + (\tilde{\sigma}^a)^2)) + \eta_t(x_t\nu_t^b + \sigma_t^\eta)^T \sigma_t^\eta
\]

The first term captures the relative risk premia that intermediaries and households earn on their portfolios relative to money. As intermediaries become undercapitalized, the price of and return from producing good \(b\) rises, leading intermediaries to take on more risk. The opposite happens when intermediaries are overcapitalized - then the price of good \(a\) and the

Figure 4: Equilibrium dynamics.

Drift of \(\eta\). The drift of \(\eta_t\) is given by

\[
\mu^\eta_t = \eta(1 - \eta)(x_t^2|\nu_t^b|^2 - (x_t^a)^2(|\nu_t^a|^2 + (\tilde{\sigma}^a)^2)) + \eta_t(x_t\nu_t^b + \sigma_t^\eta)^T \sigma_t^\eta
\]
households’ rate of earnings rises. The stochastic steady state of $\eta_t$ is the point where the drift of $\eta_t$ equals zero - at that point the earnings rates of intermediaries and households balance each other out.

The dynamics in Figure 4, together with prices and allocations as functions of $\eta$ in Figures 2 and 3 characterize the behavior of the economy in equilibrium.

### 4.2 Inefficiencies and Welfare

In this section, we calculate welfare in our model. Before we proceed, let us briefly describe the sources of inefficiency. In the process, we would like to emphasize relevant trade-offs with the intention of preparing ground for thinking about policy. First, there is inefficient sharing of idiosyncratic risk. Some of it can be mitigated through the use of intermediaries who can hold equity of households producing good $b$ and diversify some of idiosyncratic risk. Consequently, cycles that can cause intermediaries to be undercapitalized can be harmful. Inefficiencies connected with idiosyncratic risks are also mitigated with the use of money - both inside and outside. Money allows households to diversify their wealth, but high value of money results in lower price of capital and potential inefficiency due to underinvestment.

Second, there is inefficient sharing of aggregate risk, which can cause whole sectors to become undercapitalized, e.g. intermediaries. If intermediaries become undercapitalized, barriers to entry into the intermediary sector help the intermediaries: the price of good $b$ rises when $\eta_t$ is low, mitigating the intermediaries risk exposures and allowing the intermediaries to recapitalize themselves. Thus, the limited competition in the intermediary sector creates a *terms-of-trade* hedge, which depends on the extent to which intermediaries cut back the financing of households in sector $b$, the extent to which those households are willing to self-finance, and the substitutability $s$ among the intermediate goods.

Finally, there is productive inefficiency: when intermediaries or households are undercapitalized, then production may be inefficiently skewed towards good $a$ or good $b$. Even at the steady state production can be inefficient due to financial frictions, e.g. imperfect sharing of idiosyncratic risks.

To understand the cumulative effect of all these inefficiencies, one needs a proper welfare measure. Welfare analysis is complicated by heterogeneity. We cannot focus on a representative household, since different households are exposed to different idiosyncratic risks. Some households become richer, while others become poorer.

**Welfare Calculation.** Recall that, according to Proposition 3, for a general wealth
process welfare is given by (3.6). We will use this expression to calculate the welfare of intermediaries, households, as well as a fictitious “representative agent” who consumes a fixed portion of aggregate output. Intermediaries and households are the focus of our analysis, while the representative agent is a useful auxiliary construct.

**Proposition 6.** Welfare of a representative agent with net worth \( n_t = (p_t + q_t)K_t \) is given by

\[
\frac{\log(\rho n_t)}{\rho} - \frac{\log(p_t + q_t)}{\rho} + \frac{1}{\rho} E_t \left[ \int_t^\infty e^{-\rho(s-t)} \left( \log(p_s + q_s) + \Phi(t_s) - \delta - \frac{|\sigma_s^K|^2}{2} \right) ds \right].
\]  

(4.3)

**Proof.** By (3.6), the welfare of an agent who consumes \( \rho K_t \) is given by

\[
E_t \left[ \int_t^\infty e^{-\rho(s-t)} \log(\rho K_s) ds \right] = \frac{\log(\rho K_t)}{\rho} + \frac{1}{\rho} E_t \left[ \int_t^\infty e^{-\rho(s-t)} \left( \Phi(t_s) - \delta - \frac{|\sigma_s^K|^2}{2} \right) ds \right].
\]

Since \( \log(\rho(p_s + q_s)K_s) = \log(\rho K_s) + \log(p_s + q_s) \), we find that the utility of a representative agent with net worth \( \rho(p_t + q_t)K_t \) is given by (4.3).

Besides being an interesting benchmark, as a welfare measure that excludes the effects of idiosyncratic risk, measure (4.3) can be adjusted to quantify the welfare of intermediaries and households.

**Proposition 7.** The welfare of an intermediary with wealth \( n^I_t \) is \( \log(\rho n^I_t)/\rho + U^I(\eta_t) \), where

\[
U^I(\eta_t) = -\frac{\log(\eta_t(p_t + q_t))}{\rho} + \frac{1}{\rho} E_t \left[ \int_t^\infty e^{-\rho(s-t)} \left( \log(\eta_s(p_s + q_s)) + \Phi(t_s) - \delta - \frac{|\sigma_s^K|^2}{2} \right) ds \right].
\]

The welfare of a household with net worth \( n^H_t \) is \( \log(\rho n^H_t)/\rho + U^H(\eta) \), where

\[
U^H(\eta) = -\frac{\log(p_t + q_t)}{\rho} + \frac{1}{\rho} E_t \left[ \int_t^\infty e^{-\rho(s-t)} \left( \log(p_s + q_s) + \Phi(t_s) - \delta - \frac{|\sigma_s^K|^2}{2} \right) ds \right] + \frac{1}{\rho} E_t \left[ \int_t^\infty e^{-\rho(s-t)} \left( \eta_s ((x_s^a)^2(\nu_s^a)^2 + (\tilde{\sigma}_s^a)^2) - x_s^2(\nu_s^b)^2 + \frac{|\sigma_s^a|^2 - (x_s^a)^2(\nu_s^a)^2 + (\tilde{\sigma}_s^a)^2}{2} \right) ds \right].
\]
Proof. Since intermediary with net worth \( n_t = \eta_t(p_t + q_t)K_t \) consumes

\[
\log(\rho \eta_s(p_s + q_s)K_s) = \log(\eta_s) + \log(\rho(p_s + q_s)K_s),
\]

receiving the same utility flow as a representative household plus \( \log(\eta_s) \), we can obtain the welfare of an intermediary from (4.3) and obtain

\[
\frac{\log(\rho K_t)}{\rho} + \frac{1}{\rho} E_t \left[ \int_t^\infty e^{-\rho(s-t)} \left( \log(\eta_s(p_s + q_s)) + \Phi(\iota_s) - \delta - \frac{|\sigma^K_s|^2}{2} \right) ds \right].
\]

Since \( \log(\rho K_t) = \log(\rho n_t) - \log(\eta_t(p_t + q_t)) \), we obtain the desired expression.

To compute the welfare of households, notice that by Proposition 3, if two agents have wealth processes

\[
\frac{dn_t}{n_t} = \mu^n_t dt + \sigma^n_t dZ_t \quad \text{and} \quad \frac{dn'_t}{n'_t} = \mu'^n_t dt + \sigma'^n_t dZ_t,
\]

then the difference in their utility is

\[
\log(\rho n'_t) - \log(\rho n_t) - \frac{1}{\rho} E_t \left[ \int_t^\infty e^{-\rho(s-t)} \left( \mu'^n_s - \mu^n_s + \frac{|\sigma^n_s|^2 - |\sigma'^n_s|^2}{2} \right) ds \right] \quad (4.4)
\]

We can now obtain household utility by adjusting the utility of a representative agent. Recall that households are indifferent between technologies \( a \) and \( b \), so we can focus on households who use technology \( a \) without loss of generality. According to (2.17), world wealth follows

\[
\frac{dn_t}{n_t} = dr^M_t - \rho dt - (\sigma^T_t)(\sigma^M_t dt + dZ_t) + \eta_t x^2_t |\nu^b_t|^2 dt + (1 - \eta_t)(x^a_t)^2 (|\nu^a_t|^2 + (\tilde{\sigma}^a)^2) dt,
\]

while the net worth of a household that uses technology \( a \) follows

\[
\frac{dn^H_t}{n^H_t} = dr^M_t - \rho dt + x^a_t (\nu^a_t)^T (\sigma^N^a dt + dZ_t) + x^a_t (\tilde{\sigma}^a)^2 dt + \tilde{\sigma}^a d\tilde{Z}_t.
\]

Hence,

\[
\mu^H_t - \mu^a_t = \eta_t ((x^a_t)^2 (|\nu^a_t|^2 + (\tilde{\sigma}^a)^2) - x^2_t |\nu^b_t|^2) + (x^a_t \nu^a_t + \sigma^T_t \sigma^M_t).
\]
\[ \frac{|\sigma_n^a|^2 - |\sigma_{nH}^a|^2}{2} = \frac{|\sigma_t^M - \sigma_t^\theta|^2 - |\sigma_t^M + x_t^a\nu_t^a|^2 - (x_t^a)^2(\tilde{\sigma}^a)^2}{2} \] and so

\[ \mu_t^{nH} - \mu_t^n + \frac{|\sigma_s^a|^2 - |\sigma_{sH}^a|^2}{2} = \eta_t ((x_t^a)^2(|\nu_t^a|^2 + (\tilde{\sigma}^a)^2) - x_t^2|\nu_t^a|^2) + \frac{|\sigma_t^\theta|^2 - (x_t^a)^2(|\nu_t^a|^2 + (\tilde{\sigma}^a)^2)}{2} \]

Thus, using (4.3) and (4.4) to value the welfare of households, we obtain the desired expression.

This completes the derivation of the relevant welfare formulae. To then actually compute intermediary and household welfare, it suffices to note that all included quantities are functions of the single state variable \( \eta_t \), and that in general

\[ g(\eta_t) = E_t \left[ \int_t^\infty e^{-\rho(s-t)}y(\eta_s) \, ds \right] \Rightarrow \rho g(\eta) = y(\eta) + g'(\eta)\mu_t^n\eta + \frac{g''(\eta)|\eta\sigma_t^n|^2}{2} \]

The actual computation of welfare levels thus merely requires us to solve an ordinary differential equation.

**Welfare in equilibrium and preliminary thoughts on policy.** Figure 5 shows welfare for parameter values we described at the beginning of this section. For an economy with \( K_0 \) normalized to 1, Figure 5 shows the utility of a representative intermediary and a representative household (normalizing wealth dispersion among households to 0). Recall from Proposition 7 that welfare takes the form

\[ U^I(\eta_0) + \log(\rho n_0)/\rho \]

for intermediaries and

\[ U^H(\eta_0) + \log(\rho n_0^H)/\rho \]

for households.

The welfare of each agent type tends to increase in its wealth share, but only to a certain point. At the extreme, one class of agents becomes so severely undercapitalized that productive inefficiency makes everybody worse off - at those extremes redistribution towards the undercapitalized sector would be Pareto improving. Total welfare is maximized near the steady state of the system, but this property depends on the parameters we chose.

In the next section we discuss policy. Our primary focus is monetary policy, but we also look at combinations of monetary and macroprudential policies. Before proceeding, let us reiterate the inefficiencies present in our model, and discuss how policies may affect these inefficiencies. First, as in the benchmark without intermediaries, there is the trade-off between the value of money - the benefits it brings of helping hedge idiosyncratic risk - and investment distortions that arise from the control of the value of money. Of course,
monetary policy alone affects the value of money is only endogenously, while macroprudential policy can influence the value of money directly. Second, there are inefficiencies with respect to the sharing of aggregate risk - inefficiencies accompanied by production and investment distortions when one of the sectors is undercapitalized. Monetary policy can redistribute risk, and so it can help in this regard. We will also see that under monetary policy alone, risk premia, which determine earnings, are determined by the concentration of risk. Thus, monetary policy can affect the rents that each sector earns, sometimes in a way detrimental to welfare. In contrast, macroprudential policy, through its control of quantities, also has power in this regard.

5 Monetary and Macro-prudential Policy

Policy has the potential to mitigate some of the inefficiencies that arise in equilibrium. It can undo some of the endogenous risk by redistributing wealth towards compromised sectors. It can control the creation of endogenous risk by affecting the path of deleveraging. It can also work to prevent the build-up of systemic risk in booms.
Policies affect the equilibrium in a number of ways, and can have unintended consequences. Interesting questions include: What is the effect on equilibrium leverage? Does policy create moral hazard? Does policy lead to inflated asset prices in booms? What happens to endogenous risk? How does the policy affect the frequency of crises, i.e. episodes characterized by resource misallocation and loss of productivity?

We focus on several monetary policies in this section. These policies can be divided in several categories. Traditional monetary policy sets the short-term interest rate. It affects the yield curve through the expectation of future interest rates, as well as through the expected path of the economy, accounting for the supply and demand of credit, and risk premia. When the zero lower bound for the short-term policy rate becomes a constraint, forward guidance is an additional policy tool employed in practice. The use of this tool depends on central bank’s credibility, as it ties the central bank’s hands in the future and leaves it less room for discretion. In this paper we assume that the central bank can perfectly commit to contingent future monetary policy and hence the interest rate policy incorporates some state-contingent forward guidance.

Several non-conventional policies have also been employed. The central bank can directly purchases assets to support prices or affect the shape of the yield curve. The central bank can lend to financial institutions, and choose acceptable collateral as well as margin requirements and interest rates. Some of these programs work by transferring tail risk to the central bank, as it suffers losses (and consequently redistributes them to other agents) in the event that the value of collateral becomes insufficient and the counterparty defaults. Other policies include direct equity infusions into troubled institutions. Monetary policy tools are closely linked to macroprudential tools, which involve capital requirements and loan-to-value ratios.

The classic “helicopter drop of money” has in reality a strong fiscal component as money is typically paid out via a tax rebate. Importantly, the helicopter drop also has redistributive effects. As the money supply expands, the nominal liability of financial intermediaries and hence the household’s nominal savings are diluted. The redistributive effects are even stronger if the additional money supply is not equally distributed among the population but targeted to specific impaired (sub)sectors in the economy.

Instead of analyzing fiscal policy, we focus this paper on conventional and non-conventional monetary policy. For example, a change in the short-term policy interest rate redistributes wealth through the prices of nominal long-term assets. The redistributive effects of monetary policy depend on who holds these assets.\footnote{Brunnermeier and Sannikov (2012) discuss the redistributive effects in a setting in which several sectors'}
ipation of future policy, as well as the demand for insurance. Specifically, we introduce a perpetual long-term bond, and allow the monetary authority to both set the interest rate on short-term money, and affect the composition of outstanding government liabilities (money and long-term bonds) through open-market operations.

5.1 Introducing Interest Bearing Reserves/Outside Money

So far, outside money – which we can think of as reserves held by the intermediary sector – did not pay any nominal interest rate, and the total supply of outside money was fixed. From the basic Fisher equation,\(^{12}\)

\[
\frac{dM_t}{M_t} = di_t - d\pi_t,
\]

we see that the real rate (the return on money) \(d\pi_t^M\) simply corresponded to the negative of inflation, \(d\pi_t\). Indeed, given the fixed supply of outside money, economic growth – growth in \(K_t\) – led to deflation. Furthermore, whenever \(\eta_t\) declines, i.e. whenever the intermediary sector has suffered losses and is constrained in its ability to take on risk, deflation becomes even more pronounced.

In this section, we will allow for more general monetary policies. Let \(M_t\) denote the total supply of outside money. In general, \(dM_t/M_t\) can follow any stochastic process which may include possible jumps.

For most part of this paper, we assume that the central bank “prints” new outside money simply to pay the interest on existing stock of outside money, i.e. \(dM_t/M_t = di_t\). This assumes that newly printed money is distributed to existing holders of outside money (reserves). Note that nowadays most advance-economy central banks, including the U.S. Federal Reserve, pay (nominal) interest on reserves. Since intermediary sector is competitive in our model, a higher interest rate on outside money is passed on to inside money holders as well. Outside money and inside money remain to have the same return and risk profile.

Price flexibility ensures that any monetary policy rule leaves \(\eta_t\) unaffected in our simple setting. However, monetary policy evidently does have an effect on inflation. The Fisher balance sheets can be impaired. Forward guidance not to increase the policy interest rate in the near future has different implications than a further interest rate cut, since the former narrows the term spread while the latter widens it.

\(^{12}\)We write \(d\pi\) instead of \(\pi\) to capture the possibility that the price level might jump. In most of the paper we do not, in fact, allow for such jumps; the analysis of money supply rules in this section is an exception.
equation, reveals that inflation has two components: a purely monetary component related to the money supply growth rule, and a real component that reflects the various amplification spirals discussed in the previous section (which one could coin “real inflation”).

\[ d\pi_t = di_t - dr^M_t = \frac{dM_t}{M_t} - dr^M_t = \frac{dM_t}{M_t} - \frac{d(p_tK_t)}{p_tK_t} \]

We summarize our results as follows.

**Proposition 8.** (Super-Neutrality of Money) *Suppose that outside money follows the growth rule, then the analysis of Section 4 goes through unaffected, with the single difference that inflation now is*

\[ d\pi_t = \frac{dM_t}{M_t} - \frac{d(p_tK_t)}{p_tK_t}. \]

*In particular, the law of motion and stationary distribution of \( \eta_t \) are unaffected.*

We see that inflation has two components:

Our super-neutrality result is relevant for the old money view vs. credit view debate. The money view, which can be traced back to Friedman and Schwartz’s monetarism and even to the work of Irving Fisher on the Great Depression, posits that replacing in times of crisis the “missing inside money” with additional outside money suffices to stabilize the economy. This result fails to hold in our model – distribution-neutral money printing reduces disinflation, but does nothing to change real allocations. Thus, in our model, simple inflation targeting does not reduce any amplification spirals and so does not help stabilize the economy. Also, within the set of money supply rules that we consider, the Friedman rule has no special place. The Friedman rule recommends for settings in which money pays no interest a deflation rate that equalizes the real return of money with the real risk-free rate. In our setting with interest bearing reserves and flexible prices, the Friedman rule – just like all the other money supply rules – has no real effects, and so certainly is not strictly optimal within the restricted set of money supply growth rules.

The credit view, pushed primarily by Tobin, stresses the importance of restoring bank lending, and so is more concerned with the asset side of intermediaries’ balance sheets. Monetary policy that redistributes towards intermediaries can switch off amplification spirals and so improve the risk-bearing capacity of the financial sector. More generally, a monetary policy that recapitalizes balance sheet-impaired sectors can help stabilize the economy and, in
fact, may well be welfare-improving. In practice, monetary authorities do not provide *out-right* redistribution between sectors – this is the domain of fiscal policy. However, as we will illustrate in the section, both conventional and unconventional monetary policy measures (ranging from simple nominal interest rate movements to central bank purchases of long-term assets) may well indirectly (“sneakily”) re-distribute wealth between sectors. In the monetary policy can provide the desired stabilization – all in spite of complete price flexibility.

A couple of final remarks are in order. First, we need to emphasize that our negative results regarding the money view apply to money supply policies specifically designed to keep $\eta_t$ fixed. As soon as the newly printed money is distributed in a way somehow different from the existing money holdings or not passed on to inside money holders, wealth shares are impacted, and so we have real implications. For example, the classic “helicopter drop of money” would have redistributive effects. As the money supply expands, the nominal liability of financial intermediaries and hence the household’s nominal savings are diluted. The redistributive effects are even stronger if the additional money supply is not equally distributed among the population but targeted to specific impaired (sub)sectors in the economy. Thus, as soon as the newly printed money is distributed in a way somehow different from the existing (outside and inside) money holdings, wealth shares are impacted, and so we have real implications. We conclude that, for the money view to be revived in our model, monetary intervention needs to take on an explicit redistributioonal dimension. This calls for a richer modeling environment, which we will provide in the next section. Second, note that our results are derived under the assumption of complete price flexibility. Our emphasis on the credit view comes out naturally in a model with financial frictions but no price-setting frictions. Of course, matters will look different in a model with price rigidity but no financial frictions.

### 5.2 Introducing Nominal Long-term Bonds

**Money and Long-Term Bonds.** We now extend our baseline model to allow for a realistic monetary policy with distributional – and so real – effects. The extended model differs from the baseline model along two dimensions. First, as in our earlier discussion of non-redistributioonal monetary policy, we allow money/reserves to pay a floating rate of interest $i_t$, set freely by the central bank. Second, we introduce perpetual bonds that can be issued by the government. This nominal bond pays interest at a fixed rate $i^B$ in money.
The monetary authority sets the total quantity outstanding of (nominal) perpetual bonds $B_t$ (quantitative easing, or QE in short). We restrict both policies to be revenue neutral – the monetary authority pays interest and/or performs QE operations so that there are no fiscal implications. In other words, the central bank does not alter its seignorage income when changing its monetary policy.

**Interest rate policy.** First consider interest rate policy – the conventional monetary policy. To fix ideas, let us trace the implications of an interest rate cut. Cutting the short-term rate on money means that the central bank pays less money on the existing outside money stock. This then lowers the growth rate of outside money $dM_t/M_t$. Since banking is competitive, this decrease in interest paid on outside money is passed through one-to-one to interest payment on inside money. The direct effect of this interest rate cut is, just as we have seen above in the model without a long-term government bond, disinflationary. Now, however, there is an important indirect effect: Since the long-term bond continues to pay interest at a fixed nominal rate $i^B$, the price of the nominal bond relative to nominal money increases. Furthermore, as we will see later, under standard monetary policy specifications the perpetual bonds are held exclusively by the intermediaries. In short, intermediaries are long perpetual bonds and short inside money. As the price of bonds relative to the price of money rises, the intermediaries’ wealth share $\eta$ increases. For a low $\eta$ the liquidity and disinflationary spirals are thus mitigated, so that overall the second indirect effect of the rate cut is, in fact, inflationary. Algebraically, this can be seen in equation (5.1):

$$dM_t/M_t \text{ falls} - \text{the direct effect} - \text{but } d(-p_tK_t)/p_tK_t \text{ rises} - \text{the indirect effect.}$$

Overall, for the indirect effect to dominate, the response of bond prices needs to be sufficiently strong. As we will see later, a sufficient statistic for this response is the elasticity $\frac{B'(\eta)}{B/\eta}$, where $B$ denotes the nominal price of a single long-term bond in terms of money. As long as the central bank has perfect commitment power - its forward guidance is credible, this elasticity can be made arbitrarily large. For example, if the central bank were to commit to set the short-term interest rate to zero forever (say when $\eta$ drops below a certain threshold) then the relative price of the bond $B_t$ tends to infinity. Thus, around this threshold level the elasticity is very high.

These results relate to the recent debate on the interpretation of the Fisher equation and, more broadly, the relationship between nominal interest rates and inflation. Do high interest rates beget high or low inflation (and vice-versa for low interest rates)? In the
model without long-term bonds – like in all models in which money is superneutral –, for simple money supply growth rules, high interest rates beget high inflation, consistent with the recent, Fisher equation-based re-interpretation on the inflation/interest rate linkages. Conversely, as soon as interest rate movements are associated with stabilizing changes in wealth shares, the traditional short-run monetary policy view is restored. Interest rate cuts that re-distribute towards the balance sheet-impaired financial sector stabilize the economy and so tend to push up inflation.

Asset purchase programs/Quantitative Easing (QE). The implications of quantitative easing for the evolution of total outside money are more subtle. Instantaneously, QE is financed through money issuance, so central bank purchases of long-term bonds go together with increases in (short-term) outside money. But since the public is then left with fewer long-term bonds, total nominal interest payments over time are lower than before, pushing down the rate of increase of outside money \(dM_t/M_t\). At the same time, QE means that the price of the remaining long-term bonds relative to money rises, so again intermediaries are recapitalized. Thus the indirect inflationary effect of interest rate cuts is also present after expansionary asset purchase programs.\(^{13}\) At first glance, one might think that households could simply undo the central bank’s QE by adjusting their portfolio. Indeed, Wallace (1981) derived this Modigliani-Miller type result: Under the assumption that all investors can in theory purchase and short-sell arbitrary quantities of all assets at the given market prices, QE has no real effects. In contrast, in our model, and consistent with reality, households cannot issue long-term bonds, and so Wallace neutrality breaks down. Again, the strength of these effects can be summarized via the sufficient statistic \(B'(\eta)/B/\eta\).

Since the elasticity \(B'(\eta)/B/\eta\) features prominently in the analysis of both policies, we will from now on merge the presentation of the two policy examples and, more abstractly, consider the real price of bonds as a policy instrument.

Formal Analysis. For our formal analysis we denote by \(p_tK_t\) the real value of all

\(^{13}\)In reality, central bank asset purchases have not been restricted to long-term government bonds, but also included risky private sector assets. In our model, this could mean risky claims held by the intermediary sector. Whether or not such purchases would help stimulate our economy depends crucially on whether the central bank can, just like intermediaries, provide a “diversification service”. If the central bank is as bad as households in diversifying away idiosyncratic risk, then the asset purchases are ineffective. In contrast, if the central bank is as good as the intermediary sector, then it should just take over. In reality, the truth is likely to lie somewhere in between. We leave this extension for further research.
outstanding nominal assets: outside money and perpetual bonds. We denote $M_t$ the real value of (short-term) outside money and let $b_t K_t$ be the real value of all outstanding perpetual bonds. $b_t K_t$ is affected by total supply of outstanding number of bonds $B_t$ and real value of the bonds. Each individual bond’s nominal value is endogenous in equilibrium and denoted by $B_t$. We postulate that $B_t$ follows the following endogenous equilibrium process:

$$\frac{d B_t}{B_t} = \mu_t^B dt + (\sigma_t^B)^T dZ_t. \quad (5.1)$$

The real value of the perpetual bond is $M_t B_t$.

Note that the central bank chooses the nominal interest rate $i_t$ paid on short-term outside money and the number of outstanding perpetual nominal bonds $B_t = (b_t K_t) / (M_t B_t)$. To simplify the analysis, we do not treat $(i_t, B_t)$ as the policy tool pair of the central bank, but instead conjecture that it can equivalently control $b_t$ and so consider the (dual) policy instrument pair $(i_t, b_t)$. In the analysis, it then simply remains to take into account the fact that $B_t$ still evolves endogenously.

**Returns.** The expressions for the return on capital from Section 2 do not change, but money earns the return that depends on policy. If an agent holds all nominal assets in the economy - bonds and outside money - the return is

$$(\Phi(\nu) - \delta + \mu_t^p + (\sigma_t^p)^T \sigma_t^K) dt + (\sigma_t^K + \sigma_t^p)^T dZ_t.$$  

This is the return on a portfolio with weights $b_t / p_t$ on bonds and $1 - b_t / p_t$ on money. To isolate the returns on money and bonds, consider a strategy that buys bonds to earn $dr_t^B$ by borrowing money, paying $dr_t^M$. We can find the payoff of this strategy by focusing on the value of bonds in money. Using Ito’s lemma,

$$dr_t^B - dr_t^M = \left( \frac{1}{B_t} - i_t + \mu_t^B + (\sigma_t^B)^T \sigma_t^M \right) dt + (\sigma_t^B)^T dZ_t,$$

where $\sigma_t^M$ is the risk of money, which satisfies

$$\sigma_t^K + \sigma_t^p = \sigma_t^M + \frac{b_t}{p_t} \sigma_t^B \quad \Rightarrow \quad \sigma_t^M = \sigma_t^K + \sigma_t^p - \frac{b_t}{p_t} \sigma_t^B. \quad (5.2)$$
Thus, money earns the return of

\[ dr_t^M = (\Phi(\nu) - \delta + \mu_t + (\sigma_t^\nu)^T \sigma_t^K) \, dt - \frac{b_t}{p_t} \left( \frac{1}{B_t} - i_t + \mu_t^B + (\sigma_t^B)^T \sigma_t^M \right) \, dt + \sigma_t^M \, dZ_t. \] (5.3)

**Equilibrium Conditions.** For expositional purposes, we focus on policies that set the short-term interest rate \( i_t \) and the value of bonds \( b_t \) as functions of the state variable \( \eta_t \). Then the price of bonds \( B_t \) will also be a function of \( \eta_t \). Consider for concreteness policies that lead to a decreasing function \( B(\eta) \), which follows from policies that cut the short-term interest rate when \( \eta_t \) is low, making bonds appreciate. Such a policy is designed to help intermediaries transfer some of the aggregate risk to households - by borrowing money and buying long-term bonds, intermediaries get a natural hedge that gives them insurance in the event that \( \eta_t \) drops and the entire intermediary sector suffers losses. The appreciation in bonds can offset partially other risks that the intermediaries face, including endogenous risks driven by amplification. In the equations below, we assume that intermediaries hold all long-term bonds as a hedge, and later verify that this is indeed the case.

To adjust for policy, we have to take into account that the risk of money, entering the expressions for \( \nu_t, \nu_t^a, \sigma_t^{Na} \) and \( \sigma_t^{Nb} \) is now given by (5.2). Furthermore, the risk of intermediaries net worth, which has to be adjusted for the bonds they hold, now takes the form

\[ \sigma_t^N = \sigma_t^M + x_t \nu_t + x_t^B \sigma_t^B, \] where \[ x_t^B = \frac{\partial_t b_t}{\eta_t p_t} \]

is the portfolio weight on bonds.

The asset pricing conditions for capital invested in technologies \( a \) and \( b \) take the same form (2.11) and (2.10), but with the adjusted values of \( \nu_t, \nu_t^a, \sigma_t^{N_a} \) and \( \sigma_t^{N_b} \).

In addition, we have a new pricing condition for bonds,

\[ \frac{1}{B_t} - i_t + \mu_t^B + (\sigma_t^B)^T \sigma_t^M \leq (\sigma_t^M)^T \left( \sigma_t^M + x_t^a \nu_t \right), \] \[ (\sigma_t^B)^T \left( \sigma_t^M + x_t^b \nu_t \right) \] \[ \sigma_t^{N_a} \] \[ \sigma_t^{N_b} \]

Note that \( \frac{1}{B_t} - i_t \) is the nominal term spread. If the long-term bond is risk-free, i.e. \( \sigma^B = 0 \), then the term spread has to equal to the depreciation rate of the bond.

The law of motion of \( \eta_t \), likewise, has to be adjusted for the risk that the intermediaries are now exposed to.
Proposition 9. The equilibrium law of motion of $\eta_t$ is given by

$$
\frac{d\eta_t}{\eta_t} = (1 - \eta_t) \left( |x_t \nu_t^b + x_t^B \sigma_t^B|^2 - (x_t^a)^2 (|\nu_t^a|^2 + (\bar{\sigma}^a)^2) \right) dt + \left( x_t \nu_t^b + \sigma_t^\vartheta + (1 - \eta_t) x_t^B \sigma_t^B \right)^T \left( dZ_t + (\sigma_t^\vartheta - \eta_t x_t^B \sigma_t^B) dt \right).
$$

(5.4)

Proof. See Appendix. \qed

Mitigated Liquidity and Disinflationary Spiral. The effect of monetary policy on the asset-pricing conditions as well as the law of motion of $\eta_t$ enters exclusively through the term $(b_t/p_t)\sigma_t^B$. If monetary policy sets the short-term interest rate $i_t$ as well as the level of $b_t$ as functions of $\eta_t$, then the risk of the bond price $\sigma_t^B$ is collinear with $\sigma^a 1^a - \sigma_t^K$, $\sigma_t^\vartheta$ and $\sigma_t^\varphi$. Thus, monetary policy can be used implement more efficient sharing of aggregate risk, e.g. undo endogenous risk.

For example, when intermediaries borrow money to invest in equity of projects $b$, the incremental risk they take is given by

$$
\nu_t = \left( 1 - \frac{\varphi'(\eta)}{\varphi(\eta)(1 - \varphi(\eta))} + \frac{b(\eta) \ B'(\eta)}{p(\eta) \ B(\eta)} \right) (\sigma^a 1^a - \sigma_t^K). \tag{5.5}
$$

The alternative candidate is

$$
\nu_t = \sigma^b 1^b - \sigma_t^K - \frac{1}{1 - \varphi(\eta_t)/\eta_t} \frac{\varphi'(\eta_t)}{\varphi(\eta_t)/\eta_t} \sigma_t^\vartheta + \frac{b(\eta_t) \ B'(\eta_t)}{p(\eta_t) \ B(\eta_t)/\eta_t} \sigma_t^\varphi.
$$

In this equation $\frac{\varphi'(\eta)}{\varphi(\eta)(1 - \varphi(\eta))}$ reflects the amplification of aggregate risk $\sigma^a 1^a - \sigma_t^K$ through the changing prices $p_t$ and $q_t$, while $\frac{b(\eta) \ B'(\eta)}{p(\eta) \ B(\eta)}$ reflects the mitigation of this risk through monetary policy. In the following section, we study the risk transfer effects of monetary policy by focusing on the mitigating term. The one-dimensional function $\frac{b(\eta) \ B'(\eta)}{p(\eta) \ B(\eta)}$ of $\eta$ summarizes the effects of two policy tools $i_t$ and $b_t$, with which any such function can be implemented in multiple ways.
It is instructive to re-write the expression for the volatility of \( \eta \) as follows:

\[
\sigma^\eta_t = \frac{x_t(1^b \sigma^b - 1^a \sigma^b)}{1 - \left(1 - \frac{\chi_t \psi_t}{\eta_t}\right) \frac{\varphi'(\eta_t)}{\varphi(\eta_t)/\eta_t}} + \frac{b(\eta_t) - B'(\eta_t)}{p(\eta_t) B(\eta_t)/\eta_t} \left(1 - \left(1 - \frac{\chi_t \psi_t}{\eta_t} + \frac{\varphi(\eta_t)}{\eta_t}\right) \frac{1 - \eta_t}{\eta_t} \right)
\]

This expression makes clear how policy can push down endogenous risk. Indeed, as the elasticity \(-B'(\eta)/(B(\eta)/\eta)\) increases, \(\sigma^\eta\) declines and in the limit \(\eta\) becomes deterministic.

### 5.2.1 Removing Amplification

Consider a policy that sets

\[
\frac{b_t}{p_t} \sigma^B_t = \frac{\sigma^\theta}{1 - \psi_t},
\]

thus effectively removing endogenous risk in (5.5).\(^{14}\) Endogenous amplification is completely switched off and so intermediaries are only exposed to exogenous fundamental risk.

![Figure 6: Equilibrium prices of capital and money without policy (solid) and with (dashed).](image)

The following figures illustrate the application this policy to an environment with the same parameters as in Section 4, i.e. i.e. \( \rho = 0.05, A = 0.5 \sigma^a = \sigma^b = 0.4, \tilde{\sigma}^a = 1.2, \tilde{\sigma}^b = 0.6, \)

\(^{14}\)Recall that by Ito’s lemma \( \sigma^B_t = \frac{B'(\eta_t)}{B(\eta_t)/\eta_t} \sigma^\eta_t \) and similarly \( \sigma^\phi_t = \frac{\varphi'(\eta_t)}{\varphi(\eta_t)/\eta_t} \eta_t \sigma^\eta_t. \)
\[ s = 0.8, \Phi(t) = \log(\kappa t + 1)/\kappa \text{ with } \kappa = 2, \text{ and } \chi = 0.001. \] Figure 6 shows the effect of policy on prices. The price of money falls since the intermediary sector creates more inside money: it does not need to absorb as much aggregate risk to do that. As a consequence, the price of capital rises - there is more demand for capital from the sector producing good \(a\), and as Figure 7, capital is shifted to sector \(a\).

![Figure 7: Equilibrium allocations without policy (solid) and with (dashed).](image)

Finally 8 shows the drift and volatility of \(\eta\) with and without policy. With policy, the intermediary net worth is lower at the steady state. Consequently, their leverage is higher.

Ultimately, monetary policy transfers risk, but it controls quantities only indirectly – the allocation of capital, the value of money relative to capital, and earnings rates of sectors \(a\) and \(b\) as well as intermediaries are endogenously determined by the allocation of risk.

### 5.2.2 Economy with Perfect Sharing of Aggregate Risk

Recall Equation (??),

\[
\sigma_t^\eta = \frac{x_t(\sigma^b_1^b - \sigma^K)}{1 - \left(\frac{\psi_t(1 - \chi_t) - \eta_t}{\eta_t}\right) - \psi_t(1 - \chi_t) - \eta_t - \frac{\psi_t(1 - \chi_t) - \eta_t}{\eta_t} + \frac{\psi_t(1 - \chi_t) - \eta_t}{\eta_t} + \frac{\psi_t(1 - \chi_t) - \eta_t}{\eta_t} + \frac{\psi_t(1 - \chi_t) - \eta_t}{\eta_t} + \frac{\psi_t(1 - \chi_t) - \eta_t}{\eta_t} + \frac{\psi_t(1 - \chi_t) - \eta_t}{\eta_t} + \frac{\psi_t(1 - \chi_t) - \eta_t}{\eta_t}}
\]
Now let the mitigation term in the denominator go to infinity, so that ultimately $\sigma_t^\eta \to 0$. We are thus in an economy where aggregate risk is shared perfectly. Indeed, this is exactly the outcome we would see if intermediaries and households could trade contracts based on systemic risk, i.e. risk of the form

$$\left(\sigma^a 1^a - \sigma^b 1^b\right)^T dZ_t$$

With all agents sharing aggregate risk perfectly, the aggregate risk exposure of both households and intermediaries is proportional to $\sigma_t^K$, and $\eta_t$, $p_t$ and $q_t$ have no volatility. Furthermore, perfect sharing of aggregate risk implies that households who produce good $a$ will retain the minimal allowed fraction of equity, $\chi$.

The following proposition characterizes the function $\vartheta(\eta)$ through a first-order differential equation, together with $\psi_t$, household leverage $x_t^a$ and $x_t^b$, price $q_t$ and the dynamics of $\eta$.

**Proposition 10.** The function $\vartheta(\eta)$ satisfies the first-order differential equation

$$\mu_t^\vartheta = \frac{\vartheta'(\eta)}{\vartheta(\eta)} \eta \mu_t^\vartheta, \quad (5.7)$$
where
\[ \mu_t^\eta = -(1 - \eta)(x_t^b)^2(\tilde{\sigma}^b)^2, \quad \mu_t^\theta = \rho + \mu_t^\eta, \] (5.8)
and \( \psi_t, x_t^a, x_t^b \) and \( q_t \) satisfy
\[ A(\psi_t) - \iota(q_t) = \frac{\rho q_t}{1 - \vartheta_t} \frac{\psi_t \chi x_t^a / \tilde{x}_t^a + 1 - \psi_t}{x_t^b} = \frac{1 - \eta_t}{1 - \vartheta_t} x_t^a \sigma_t^a = x_t^b \sigma_t^b \quad \text{and} \] (5.9)
\[ \frac{A^a(\psi_t) - A^b(\psi_t)}{q_t} = \psi_t (\sigma_t^a)^2 - (1 - \psi_t) (\sigma_t^b)^2 + \chi_x x_t^a (\tilde{\sigma}^a)^2 - x_t^b (\tilde{\sigma}^b)^2. \] (5.10)

**Proof.** See Appendix.

![Figure 9: Equilibrium with perfect sharing of aggregate risk.](image)

Figure 9 shows prices in the benchmark of perfect aggregate risk sharing for parameter values \( \rho = 5\% \), \( A = 0.5 \), \( \sigma_t^a = \sigma_t^b = 0.4 \), \( \tilde{\sigma}^a = 1.2 \), \( \tilde{\sigma}^b = 0.6 \), \( s = 0.8 \), \( \Phi(\iota) = \log(\kappa \iota + 1)/\kappa \) with \( \kappa = 2 \), and \( \chi = 0.001 \). The horizontal axis corresponds to the intermediary net worth share \( \eta_t \). Due to perfect sharing of aggregate risk, intermediaries hold all available outside equity of households who produce good \( a \), hedging risks perfectly, regardless of their net worth. As intermediary net worth rises, the net worth of households, and thus their capacity to absorb idiosyncratic risks, falls. Output falls as \( \eta_t \) rises, as seen in the left panel. The right panel shows how the prices of money and capital change with \( \eta_t \). It is noteworthy that the value
of money is very low relative to both the money regime and the full equilibrium that we describe in the next section. The value of money \( \vartheta_t \) relative to total wealth rises. The drift of \( \eta_t \) is always negative, as shown on the right panel and seen from (5.8).

In contrast, without intermediaries (3.3) implies that the prices of capital and money would be \( q = 1.0655 \) and \( p = 3.79 \) (see Lemma ? in the Appendix). The value of money is significantly higher under the benchmark without intermediaries who provide insurance against almost all of idiosyncratic risk of technology \( a \). This fact creates the possibility of a significant deflationary spiral in our full model, in which the intermediaries have to absorb some of aggregate risk, and their capacity to function depends on having sufficient net worth.
5.3 Welfare

We have to adjust the equations from Section 4.2 somewhat since the baseline return on money is given by the generalized formula (5.3).

**Proposition 11.** The welfare of an agent with wealth $n_t$ who can invest only in money is given by
\[ \omega(\eta_t) + \log(n_t/p_t)/\rho, \]
where $\omega(\eta)$ satisfies
\[ \rho \omega(\eta_t) = \log(\rho_{pt}) + \mu^\eta_t \eta \omega'(\eta) + \frac{|\sigma^\eta_t \eta|^2}{2} + \frac{\Phi(\iota) - \delta - \rho - (\sigma^K_t)^2}{\rho} - \frac{b^p_t \sigma^B_t (x_t \nu_t + \frac{b^p_t b^p_t \sigma^B_t)}{\eta_t \rho} + \frac{(b^p_t \sigma^B_t)^2}{2\rho}}. \]  
(5.11)

The welfare of an intermediary with net worth $n_t$ is given by
\[ h(\eta_t) + \omega(\eta_t) + \log(n_t/p_t)/\rho, \]
where $\rho h(\eta_t) = x^2_t |\nu_t|^2 + h^t(\eta) \mu^\eta_t \eta + \frac{h^t(\eta) \eta^2 \sigma^\eta_t (\sigma^\eta_t)^T}{2}. \]  
(5.12)

The welfare of a household is $h^J(\eta_t) + \omega(\eta_t) + \log(n_t/p_t)/\rho$, where $h^J$ satisfies equation (D.2) with the term $x^2_t |\nu_t|^2$ replaced by $(x^J_t)^2 |\nu_t|^2$.

**Proof.** From the bond-pricing condition, the return on money (5.3) can be written as
\[ (\Phi(\iota) - \delta - \rho - (\sigma^K_t)^2 - \frac{b^p_t \sigma^B_t (x_t \nu_t + \frac{b^p_t b^p_t \sigma^B_t)}{\eta_t \rho} + \frac{(b^p_t \sigma^B_t)^2}{2\rho}) dZ_t. \]

Conjecturing the form $\omega(\eta_t) + \log(n_t/p_t)/\rho$ for the value function of someone who invests only in money, the HJB equation is
\[ \rho \omega(\eta_t) + \log(n_t/p_t) = \log(\rho_{nt}) + \mu^\eta_t \eta \omega'(\eta) + \frac{|\sigma^\eta_t \eta|^2}{2} + \frac{\Phi(\iota) - \delta - \rho - (\sigma^K_t)^2}{\rho} - \frac{b^p_t \sigma^B_t (x_t \nu_t + \frac{b^p_t b^p_t \sigma^B_t)}{\eta_t \rho} + \frac{(b^p_t \sigma^B_t)^2}{2\rho}}. \]

Plugging in
\[ \sigma^N_t = \sigma^M_t + x_t \nu_t + \frac{\partial_t b_t}{\eta_t p_t} \sigma^B_t, \quad \sigma^M_t = \sigma^K_t + \sigma^p_t - \frac{b_t}{\rho p_t} \sigma^B_t, \]  
the equation for $\omega$ simplifies to (5.11). Furthermore, as in the proof of Proposition 13, the welfare of any agent with richer investment opportunities has to be adjusted by the incremental risk that the agent chooses to take. In particular, the welfare of an intermediary...
with net worth \( n_t \) is given by \( h(\eta_t) + \omega(\eta_t) + \log(n_t/p_t)/\rho \), where \( h(\eta) \) satisfies (5.12). Likewise, to get the welfare of households \( h^J(\eta_t) + \omega(\eta_t) + \log(n_t/p_t)/\rho \), we have to replace the term \( x_t^2|\nu_t|^2 \) with \( (x_t^J)^2|\nu_t^J|^2 \).

The relevant boundary conditions are the same as without policy, since the policy affects equilibrium dynamics only in the interior of the state space \([0, 1]\) and not on the boundaries.

5.4 Policies that undo Endogenous Risk.

The risk that intermediaries take on by buying more capital is captured by

\[
\nu_t = \left( 1 - \frac{\vartheta'(\eta)}{\vartheta(\eta)(1 - \vartheta(\eta))} + \frac{b_t B'(\eta)}{p_t B(\eta)} \right) \psi_t^J \left( \frac{\sigma^I 1^I}{I} - \frac{\sigma^J 1^J}{J} \right) = (1 + \alpha_t) \psi_t^J \left( \frac{\sigma^I 1^I}{I} - \frac{\sigma^J 1^J}{J} \right)
\]

where the term \( \frac{-\vartheta'(\eta)}{\vartheta(\eta)(1 - \vartheta(\eta))} > 0 \) amplifies the risk that intermediaries face and \( \frac{b_t B'(\eta)}{p_t B(\eta)} \) can mitigate it.

Likewise, the risk that households take on by buying capital is

\[
\nu_t^J = (1^J \sigma^J - \sigma^K) + \left( -\frac{\vartheta'(\eta)}{\vartheta(\eta)(1 - \vartheta(\eta))} + \frac{b_t B'(\eta)}{p_t B(\eta)} \right) \psi_t^J \left( \frac{\sigma^I 1^I}{I} - \frac{\sigma^J 1^J}{J} \right),
\]

which consists of idiosyncratic risk and average \( J \)-household risk

\[
\nu_t^J = -(1 - \psi_t^J) \left( \frac{\sigma^I 1^I}{I} - \frac{\sigma^J 1^J}{J} \right) + \alpha_t \psi_t^J \left( \frac{\sigma^I 1^I}{I} - \frac{\sigma^J 1^J}{J} \right) = (\alpha_t \psi_t^J - (1 - \psi_t^J)) \left( \frac{\sigma^I 1^I}{I} - \frac{\sigma^J 1^J}{J} \right)
\]

It looks like \( \alpha_t > 0 \) in the absence of monetary policy, and monetary policy can reduce \( \alpha_t \). Effectively, this action shifts risk between technologies \( I \) and \( J \) - a reduction in \( \alpha \) shifts risk in favor of \( I \). Policy affects the world portfolio, and this (1) the total output/productive efficiency, (2) the amount of idiosyncratic risk exposure.

The agents’ risk exposures are given by

\[
\sigma_t^K + \sigma_t^q - \alpha_t \psi_t^J \left( \frac{\sigma^I 1^I}{I} - \frac{\sigma^J 1^J}{J} \right) + x_t^J (\alpha_t \psi_t^J - (1 - \psi_t^J)) \left( \frac{\sigma^I 1^I}{I} - \frac{\sigma^J 1^J}{J} \right) + x_t^J \, d\epsilon_t^J.
\]
by which fundamental risk is amplified. We observe that amplification is significant, particularly below the steady state when intermediaries are undercapitalized.

In this section we consider the effect of policy that completely removes endogenous risk by setting instruments $b_t$ and $r_t$ in such a way that

$$\frac{\sigma^\theta}{1-\vartheta} = \sigma^p_t - \sigma^q_t = \frac{b_t}{p_t} \sigma^B_t. \quad (5.13)$$

This can be done in multiple ways, since we have the flexibility to choose two functions of $\eta$ to match a single condition (5.13). If (5.13) holds, then

$$\nu_t = \sigma^I_t \vartheta - \sigma^K_t \vartheta \quad \nu'_t = \sigma^i_t - \sigma^K_t \quad \text{and} \quad \nu''_t = \sigma^J_t - \sigma^K_t$$

i.e. the incremental risk that any agent faces by adding capital to his/her portfolio is only fundamental and not endogenous. In this section we show that the equilibrium dynamics that results under any such policy can be characterized in terms of a single second-order differential equation for the function $\vartheta(\eta)$.

First, the law of motion of $\eta_t$ can be found from (5.4). Given (5.13), this reduces to

$$\frac{d\eta_t}{\eta_t} = (1 - \eta_t) \left( \left| x_t \nu_t + \frac{\vartheta_t}{\eta_t} \frac{\sigma^\theta}{1-\vartheta} \right|^2 dt - |x_t \nu'_t|^2 \right) dt +$$

$$\left( \frac{1 - \eta_t}{\eta_t} \frac{\vartheta_t \sigma^\theta}{1-\vartheta_t} + x_t \nu_t \right) \left( dZ_t - \frac{\partial \sigma^\theta}{1-\vartheta} dt \right). \quad (5.14)$$

Thus,

$$\sigma^\eta_t = x_t \left( \sigma^I_t \vartheta - \sigma^K_t \right) + \frac{1 - \eta_t}{1 - \vartheta_t} \vartheta(\eta) \sigma^\eta \quad \Rightarrow \quad \sigma^\eta_t = x_t \left( \frac{\sigma^I_t \vartheta - \sigma^K_t}{1 - \frac{1 - \eta_t}{1 - \vartheta_t} \vartheta(\eta)} \right),$$

where the denominator has dampening effect as long as $\vartheta'(\eta) < 0$, since it reduces the risk of $\eta$. Second, the function $\vartheta(\eta)$ it self can be found via the following procedure.

**Procedure.** The function $\vartheta(\eta)$ that results under any policy that removes endogenous risk according to (5.13) has to satisfy the following second-order differential equation. First,
given \( \eta \) and \((\vartheta(\eta), \vartheta'(\eta))\) the allocation of capital \((\psi_t, \psi^J_t)\) must satisfy the conditions

\[
x_t = \frac{\psi_t(1 - \vartheta(\eta))}{\eta_t}, \quad \sigma^\eta_t = \frac{x_t \left(\sigma^1 \frac{1}{T} - \sigma^K_t\right)}{1 - \frac{1 - \eta}{1 - \vartheta}} \vartheta'(\eta), \quad \sigma^\vartheta_t = \frac{\vartheta'(\eta)}{\vartheta(\eta)} \sigma^\eta_t, \quad Y - \nu(q) = \frac{\rho q}{1 - \vartheta}\nonumber,\]

\[
x_t \nu_t |^2 + \nu_t \frac{\vartheta_t (\sigma^\vartheta)^T}{\eta_t 1 - \vartheta} = x_t | \nu_t |^2, \quad \frac{(P_t - P^i_t)a}{q_t} = (\sigma^J 1^i - \sigma^I 1^j)(\sigma^K_t)^T + x_t | \nu_t |^2 - x_t | \nu^J_t |^2. \quad (5.15)\nonumber\]

\[
\eta + \frac{\psi^J(1 - \vartheta(\eta))}{\eta^J} + \frac{(1 - \psi - \psi^J)(1 - \vartheta(\eta))}{\eta^J} = 1 \quad \text{and} \quad (x_t | \nu_t | ^2 \leq (x_t | \nu^J_t | ^2),
\]

with equality of \( \psi + \psi^J < 1 \). Then,

\[
\frac{P_t a - \nu}{q} + \frac{\sigma^\vartheta}{1 - \vartheta} = \nu_t \left(\sigma^\eta + \frac{\vartheta_t \sigma^\vartheta_t}{1 - \vartheta_t} + \sigma^K_t\right)^T + \frac{\mu^\vartheta}{1 - \vartheta} 
\quad (5.16)\nonumber\]

\[
\text{and} \quad \vartheta''(\eta) = \frac{\mu^\vartheta \vartheta(\eta) - \vartheta'(\eta) \mu^\eta_t \eta}{\eta^2 \sigma^\eta_t (\sigma^\eta_t)^T / 2}, \nonumber\]

where the volatility of \( \eta, \mu^\eta_t \), is taken from (5.14).

**Proof.** We only need to justify the expressions that appear in the above procedure for the first time. Note that

\[
\sigma^N_t = \sigma^K_t + \sigma^p_t - \frac{b_t}{p_t} \sigma^B_t + x_t \left(\sigma^1 \frac{1}{I} - \sigma^K_t\right) + \frac{\vartheta_t \sigma^\vartheta}{\eta_t 1 - \vartheta}, \quad \sigma^M_t = \sigma^K_t + \sigma^\eta_t
\]

Subtracting the capital pricing conditions for intermediaries and households who hold capital types \( i = 1, \ldots I, \) we obtain

\[
x_t | \nu_t |^2 + \nu_t \frac{\vartheta_t (\sigma^\vartheta)^T}{\eta_t 1 - \vartheta} = x_t | \nu_t |^2,
\]

since \( \nu_t (\sigma^M_t)^T = \nu^J_t (\sigma^M_t)^T \). Furthermore, the capital pricing conditions of intermediaries and
households who hold capital types \( j = I + 1, \ldots I + J \), we find
\[
\frac{(P_i^I - P_i^J) a_i}{q_t} + (\sigma_i^I 1^i - \sigma_i^J 1^j)(\sigma_i^q)^T = x_t^i |\nu_t^I|^2 - x_t^j |\nu_t^J|^2 + (\sigma_i^I 1^i - \sigma_i^J 1^j)(\sigma_i^K + \sigma_i^q)^T,
\]
which simplifies to the second equation in (A.1).

Finally, from the capital and bond pricing conditions for intermediaries, we obtain
\[
E[dr_t^I - dr_t^M] = \frac{P_t^I a - \mu_t^q - \mu_t^p + \sigma_t^q}{q} + \frac{\sigma_t^q}{1 - \vartheta} + \frac{\sigma_t^p}{1 - \vartheta} = \nu_t \sigma_t^N.
\]
Since
\[
\sigma_t^N = \sigma_t^q + \frac{\vartheta_t \sigma_t^q}{1 - \vartheta_t} + \sigma_t^M
\]
and
\[
\mu_t^p - \mu_t^q = \frac{\sigma_t^p}{1 - \vartheta} - \frac{(\sigma_t^q)^2}{1 - \vartheta} + \frac{\mu_t^q}{1 - \vartheta}, \quad \sigma_t^p - \sigma_t^q = \frac{\sigma_t^q}{1 - \vartheta},
\]
we have
\[
\frac{P_t^I a - \mu_t^q - \mu_t^p + \sigma_t^q}{q} + \frac{\sigma_t^q}{1 - \vartheta} = \nu_t \left( \sigma_t^q + \frac{\vartheta_t \sigma_t^q}{1 - \vartheta_t} + \sigma_t^M \right) + \frac{\mu_t^q}{1 - \vartheta},
\]
as required.

We can verify analytically that for this policy, only intermediaries choose to hold long-term bonds. The following proposition states this fact.

**Proposition 12.** Under the policy that removes endogenous risk, if \( \vartheta'(\eta) < 0 \) for all \( \eta \in (0, 1) \), then only intermediaries want to hold bonds at all points of the state space. That is, the required risk premium for bond holdings is lower for intermediaries than any other agents, i.e.
\[
\sigma_t^B(\sigma_t^N)^T \leq \sigma_t^B(\sigma_t^M + x_t^I \nu_t^I)^T, \quad \sigma_t^B(\sigma_t^M + x_t^I \nu_t^I)^T, \quad (5.17)
\]
where \( \sigma_t^N = \sigma_t^M + x_t \nu_t + \frac{\vartheta_t}{1 - \vartheta_t} \sigma_t^q \).

**Proof.** First, note that
\[
\nu_t = \psi_t \left( \sigma_t^I 1^I - \sigma_t^J 1^J \right), \quad |\nu_t|^2 = \nu_t^I \nu_t = (\psi_t^I)^2 \left( \frac{(\sigma_t^I)^2}{I} - \frac{(\sigma_t^J)^2}{J} \right)
\]
and
\[
\nu_t(\nu_t^I)^T = -(1 - \psi_t) \psi_t \left( \frac{(\sigma_t^I)^2}{I} - \frac{(\sigma_t^J)^2}{J} \right).
\]

Furthermore,

\[
\sigma_t^B = \frac{p_t}{b_t} \left( \frac{\sigma_t^\theta}{1 - \vartheta_t} \right) = \frac{p_t}{b_t} \frac{\vartheta'(\eta)}{\vartheta_t(1 - \vartheta_t)} \frac{\eta_t}{1 - \frac{n}{\vartheta_t}} \vartheta'(\eta) \nu_t.
\]

Thus, (5.17) is equivalent to

\[
\nu_t \left( x_t \nu_t + \frac{\vartheta_t}{\eta_t (1 - \vartheta_t)} \sigma^\theta \right)^T \geq x_t^I \nu_t (\nu_t^I)^T, \quad x_t^I \nu_t (\nu_t^I)^T.
\]

The first inequality holds since

\[
x_t^I \nu_t (\nu_t^I)^T < 0 \text{ and } x_t^I \nu_t (\nu_t^I)^T = x_t \nu_t (\nu_t^I)^T = x_t \nu_t (\nu_t^I)^T,
\]

so \( \nu_t \left( x_t \nu_t + \frac{\vartheta_t}{\eta_t (1 - \vartheta_t)} \sigma^\theta \right)^T > 0 \). The second inequality holds since

\[
\nu_t \left( x_t \nu_t + \frac{\vartheta_t}{\eta_t (1 - \vartheta_t)} \sigma^\theta \right)^T = x_t |\nu_t^I|^2,
\]

and \(|\nu_t^I|^2 > |\nu_t|^2\), specifically,

\[
|\nu_t^I|^2 = |\nu_t|^2 + (\sigma^\theta)^2 \frac{I - 1}{I}.
\]

Who holds the bonds... note also that the intermediaries’ greater willingness to hold bonds depends on

\[
\sigma^\theta (x \nu + \vartheta / \eta \sigma^\theta / (1 - \vartheta)) < \sigma^\theta x^J \nu^J,
\]

i.e. the intermediaries’ required risk premium for holding bonds is lower...

Implementation. The condition (5.13) requires

\[
\frac{d}{d \eta} \log \vartheta(\eta) = \frac{b_t (1 - \eta_t)}{p_t} \frac{d}{d \eta} \log B(\eta).
\]

This makes it pretty easy to solve for \( \log B(\eta) \). Since \( B \) is steeper than \( \vartheta \), the policy may need to buy access bonds when \( \eta \) is low (quantitative easing) and reverse the policy when \( \eta \)
is high.

\[ \frac{P^t_i a - \iota}{q_t} = \left( \nu_t - \frac{\sigma^\theta_t}{1 - \eta_t} \right) (\sigma_t^i + \sigma^K_t)^T + \sigma^K_t (\sigma^\varphi_t)^T + \left( \sigma^I_t \frac{1}{I} \right) \frac{\partial_t (\sigma^\varphi)^T}{1 - \eta_t} + \frac{\mu^\varphi_t}{1 - \eta_t} \]

\[ \frac{\mu^\varphi_t}{1 - \eta_t} = \frac{P^t_i a - \iota}{q_t} - (\sigma^\theta_t + \sigma^K_t) \nu_t^T - \sigma^K_t (\sigma^\varphi_t)^T - \frac{\partial_t (\sigma^\varphi)^T}{1 - \eta_t} \left( \sigma_i^I \frac{1}{I} \right)^T \]

Combining (5.18) and (5.19), we obtain \( x_i |\nu_t|^2 = x_i |\nu_i|^2 \) since \( \nu_t (\sigma^K_t + \sigma^p_t) = \nu_i (\sigma^K_i + \sigma^p_i) \).

Subtracting (5.18) and (5.19), we obtain

\[ \frac{(P^t_i - P^j_i)}{q} + (\sigma^I \nu_i - \nu_i^j) (\sigma^\varphi_i)^T = x_i |\nu_i|^2 - x_i^j |\nu_j|^2 + (\sigma^I \nu_i - \nu_i^j) (\sigma^K_i + \sigma^p_i)^T, \]

which implies the second equation in (5.19) since \( \sigma^p - \sigma^\varphi = \sigma^\varphi / (1 - \eta). \) Finally, (5.19) is expanded to

\[ \frac{P^t_i a - \iota}{q} + \mu^\varphi_t - \mu^\varphi_i + \sigma^\varphi_i \left( \sigma^I \frac{1}{I} \right)^T - \sigma^p_i (\sigma^K_t)^T = x_i \nu_i \nu_i^T + \nu_t (\sigma^K + \sigma^p_i)^T \]

(5.18)

Since

\[ \mu^\varphi_t - \mu^\varphi_i = \frac{\sigma^\varphi_t}{1 - \eta_t} - \frac{\eta_t \sigma^\varphi_i}{1 - \eta_t} + \frac{\mu^\varphi_t}{1 - \eta_t}, \]

and \( \sigma^\varphi_i = x_i \nu_i + \sigma^\varphi_t \), we have (5.18) can be transformed to

\[ \frac{P^t_i a - \iota}{q} - \frac{\mu^\varphi_t}{1 - \eta_t} = (\sigma^\varphi_t + \sigma^K_t) \nu_t^T + \sigma^\varphi_i (\sigma^K_t)^T + \sigma^I_t \frac{1}{I} \frac{\partial \sigma^\varphi_t}{1 - \eta_t}, \]

which confirms (A.2).

We have

\[ \frac{(P^t_i - P^j_i)}{q_t} (\sigma^I \nu_i - \nu_i^j) = \nu_t (\sigma^M_i + x_i \nu_t + \frac{\partial_t (\sigma^\varphi)}{\eta_t 1 - \eta_t})^T - \nu_i^j (\sigma^M_i + x_i^j \nu_j^T)^T. \]

\[ \frac{(P^i - P^j_i)}{q_t} = (\sigma_i^I - \nu_i^j) \sigma^K_i + x_i |\nu_t|^2 + \nu_t \frac{(\sigma^\varphi)^T}{\eta_t 1 - \eta_t} - x_i^j |\nu_j|^2. \]
The pricing conditions for capital and money on the intermediaries balance sheets are

\[
\frac{E_t[dr^I_t - dr^M_t]}{dt} = \nu_t(\sigma^N_t)^T \quad \text{and} \quad \frac{1}{B_t} - r_t + \mu^B_t + \sigma^B_t(\sigma^M_t)^T = \sigma^B(\sigma^N_t)^T.
\]

Likewise, the households optimal allocations to capital \(x^I_t\) and \(x^J_t\) must satisfy

\[
\frac{E_t[dr^I_t - dr^M_t]}{dt} = \nu_t(\sigma^M_t + x^I_t\nu^I_t)^T \quad \text{and} \quad \frac{E_t[dr^J_t - dr^M_t]}{dt} = \nu_t(\sigma^M_t + x^J_t\nu^J_t)^T.
\]

We must also verify that

\[
\frac{1}{B_t} - r_t + \mu^B_t + \sigma^B_t(\sigma^M_t)^T \leq \sigma^B_t(\sigma^M_t + x^I_t\nu^I_t)^T, \quad \sigma^B_t(\sigma^M_t + x^J_t\nu^J_t)^T
\]

\[
dr^I_t = \frac{P^I_t - \epsilon}{q} dt + \left(\Phi(\nu) - \delta + \mu^p_t + \sigma^p_t \left(\frac{\sigma^I_t}{I}\right)^T\right) dt + \left(\sigma^q_t + \sigma^q_t \frac{1}{I}\right) dZ_t,
\]

\[
dr^M_t = \left(\Phi(\nu) - \delta + \mu^p_t + \sigma^p_t(\sigma^K_t)^T\right) dt - \frac{\sigma^q}{1 - \eta}(\sigma^N_t)^T dt + \sigma^M_t dZ_t.
\]

Ok

\[
\frac{P^I_t - \epsilon}{q} + \mu^p_t - \mu^p_t + \sigma^q_t \left(\frac{\sigma^I_t}{I}\right)^T - \sigma^p_t(\sigma^K_t)^T + \frac{\sigma^q}{1 - \eta}(\sigma^N_t)^T = \nu_t \left(\frac{\sigma^q_t}{1 - \eta} + \nu_t \frac{\sigma^q_t}{1 - \eta} + \sigma^M_t\right)
\]

\[
\frac{P^I_t - \epsilon}{q} + \frac{\sigma^q}{1 - \eta} \sigma^q_t + \frac{\sigma^q}{1 - \eta} \frac{\partial_t \sigma^q}{1 - \eta} = \nu_t \left(\frac{\sigma^q_t}{1 - \eta} + \frac{\partial_t \sigma^q}{1 - \eta} + \sigma^K_t\right)^T + \frac{\sigma^q}{1 - \eta} \frac{\sigma^q}{1 - \eta} - \frac{(\sigma^q)^2}{1 - \eta} + \frac{\mu^q}{1 - \eta}
\]

\[
\frac{P^I_t - \epsilon}{q} + \frac{\sigma^q}{1 - \eta} \sigma^q_t = \nu_t \left(\frac{\sigma^q_t}{1 - \eta} + \frac{\partial_t \sigma^q}{1 - \eta} + \sigma^K_t\right)^T + \frac{\mu^q}{1 - \eta}
\]

\[
\frac{\mu^q}{1 - \eta} = \frac{P^I_t - \epsilon}{q} + \frac{\sigma^q}{1 - \eta} \sigma^q_t - \left(\frac{\sigma^I_t}{I} - \frac{\sigma^K_t}{1 - \eta}\right) \left(\frac{\sigma^I_t}{I} + \frac{\sigma^K_t}{1 - \eta} + \frac{\partial_t \sigma^q}{1 - \eta} \frac{1}{1 - \eta}\right)
\]

\[
\mu^p_t - \mu^q_t = \frac{\sigma^q}{1 - \eta} - \frac{(\sigma^q)^2}{1 - \eta} + \frac{\mu^q}{1 - \eta} + \frac{\sigma^q_t}{1 - \eta}, \quad \sigma^p_t - \sigma^q_t = \frac{\sigma^q}{1 - \eta},
\]
\[ \sigma^\eta_t = x_t \left( \sigma^{1I}_I - \sigma^K_t \right) + \hat{\sigma}_t \frac{1 - \eta_t - \sigma^\eta_t}{1 - \eta_t} \]

Before we had

\[ \frac{P^I_t a - \sigma^I_t a}{q} = x_t \nu_t^2 + \left( \sigma^{1I}_I - \sigma^I_t I \right) \left( \sigma^p_t - \sigma^q_t + \sigma^K_t T - x_t \nu_t^2 \right) \]

\[ \frac{P^I_t a - \sigma^I_t a}{q} = \frac{(\sigma^\eta)^2}{1 - \vartheta} + \frac{\mu^\vartheta}{1 - \vartheta} + (\sigma^\eta_t - \sigma^p_t) \left( \sigma^{1I}_I \right)^T = x_t \nu_t^2 + \left( \sigma^{1I}_I + \sigma^q_t - \sigma^p_t - \sigma^K_t \right) \sigma^K_t. \]

Another way to write it is

\[ \frac{P^I_t a - \sigma^I_t a}{q} + \left( \sigma^\eta_t \right)^2 - \frac{\mu^\vartheta}{1 - \vartheta} + (\sigma^\eta_t - \sigma^p_t) \left( \sigma^{1I}_I \right)^T = x_t \nu_t^2 + \left( \sigma^{1I}_I + \sigma^q_t - \sigma^p_t - \sigma^K_t \right) \sigma^K_t. \]

\[ \frac{P^I_t a - \sigma^I_t a}{q} - \frac{\mu^\vartheta}{1 - \vartheta} = \sigma^\eta_t \nu_t^T + \sigma^K_t \nu_t^T + \sigma^K_t \sigma^\vartheta_t + \frac{\vartheta \sigma^\eta_t}{1 - \vartheta} \left( \sigma^{1I}_I \right)^T \]

With policy,

\[ \frac{P^I_t a - \sigma^I_t a}{q} - \frac{\mu^\vartheta}{1 - \vartheta} = \sigma^\eta_t \left( \sigma^{1I}_I - \sigma^K_t \right) + \left( \sigma^{1I}_I - \sigma^K_t \right) \left( \sigma^K_t + \vartheta \frac{\sigma^\eta_t}{1 - \vartheta} \right) \]

or

\[ \frac{P^I_t a - \sigma^I_t a}{q} - \frac{\mu^\vartheta}{1 - \vartheta} = \sigma^\eta_t \nu_t^T + \sigma^K_t \nu_t^T + \sigma^K_t \left( \sigma^\vartheta_t \right)^T + \frac{\vartheta \sigma^\eta_t}{1 - \vartheta} \left( \sigma^{1I}_I \right)^T \]

i.e. an identical formula applies in both cases.

### 5.5 Monetary Policy: An Example

This section provides an example of how monetary policy can affect equilibrium dynamics and welfare. We take the same parameters as in our example in Section 3. We then focus
on the extent to which policy mitigates endogenous risk. Specifically, consider policies that lead to

\[ \frac{b_t}{p_t} \sigma_t^B = \alpha(\eta) \frac{\sigma_t^p}{1 - \psi}, \]

so

\[ \sigma_t^\eta = \frac{\sigma_t^B}{1 + (\psi - \eta)(1 - \alpha_t) \frac{\vartheta(\eta)}{\vartheta(\eta)} - \alpha(\eta) \frac{1 - \vartheta_t}{1 - \vartheta_t} \vartheta_t(\eta)} \]

and

\[ \nu_t = \left( 1 - (1 - \alpha(\eta)) \frac{\vartheta(\eta)}{\vartheta(\eta)(1 - \vartheta(\eta))} \right) \left( \sigma_t^B - \sigma_t^K \right), \]

where \( \alpha(\eta) \in [0, 1] \). In the following example, \( \alpha(\eta) = \max(0.5 - \eta, 0) \), i.e. monetary policy eliminates up to a half of endogenous risk for \( \eta < 0.5 \). Figure 10 compares several equilibrium quantities with and without policy, and zooms around the steady state to make the comparison clearer.

![Figure 10: Equilibrium with and without policy.](image-url)
The bottom right panel illustrates that policy reduces risk. As a result, intermediaries are able to employ their technologies to a greater extent even for lower levels of net worth, and households step in to inefficiently use the intermediaries’ technologies at a lower level of \( \eta_t \) - see top right panel. At the same time, intermediaries are able to maintain higher leverage - their net worth at the steady state is reduced as shown in the bottom left panel. The top left panel shows that the value of money is somewhat lower with the policy - due to the fact that intermediaries are able to function more efficiently and create more inside money. The price of capital \( q(\eta) \) does not change significantly (it becomes slightly higher with the policy).

![Diagram](image.png)

**Figure 11: Welfare in Equilibrium.**

Figure 11 confirms that this policy, though appropriate risk transfer, indeed improves welfare. It replicates Figure 5, and compares welfare measures with and without policy. Overall welfare clearly improves with policy, as we can see from the right panel of Figure 11. Moreover, total welfare is maximized at a lower level of \( \eta \), so that the shift of the steady state to the left is not detrimental. We estimated that this improvement in welfare is equivalent to increased consumption by about 9% per year. The left panel shows the effect on individual agents. Households get slightly lower utility given any wealth level with policy, but also at the steady state they have greater wealth.
Compare with

\[ \rho \omega(\eta) = \log(\rho p(\eta)) + \frac{\Phi(\iota(\eta)) - \delta - \rho - \sigma^2/2}{\rho} + \omega'(\eta)\mu_n^\eta \eta + \frac{\omega''(\eta)\eta^2\sigma_n^\eta(\sigma_n^\eta)^T}{2}. \]

Let \( H_0(\eta) = \omega(\eta_t) - \log(p_t)/\rho \). Then this agent’s utility can be also expressed as \( H_0(\eta_t) + \log(n_t)/\rho \), where \( H_0 \) satisfies the HJB equation

\[ \rho H_0(\eta_t) = \log(\rho) + \frac{1}{\rho} E \left[ \frac{dn_t}{n_t} \right] / dt - \frac{(\sigma_t^p + \sigma_t^K)^2}{2\rho} + H'_0(\eta_t)\mu_n^\eta \eta + \frac{H''_0(\eta_t)\eta^2\sigma_n^\eta(\sigma_n^\eta)^T}{2}. \] (5.19)

Now, consider another agent whose best investment opportunity over money has risk \( \nu_t \) and who (given the returns) chooses to put portfolio weight \( x_t \) on this opportunity. Then this agent’s wealth follows

\[ \frac{d\tilde{n}_t}{\tilde{n}_t} = \frac{dn_t}{n_t} + x_t\nu_t(x_t\nu_t + \sigma_t^p + \sigma_t^K)^T dt + x_t\nu_t dZ_t. \]

The utility of this agent can be represented in the form \( H(\eta_t) + \log(\tilde{n}_t)/\rho \), where

\[ \rho H(\eta_t) = \log(\rho) + \frac{1}{\rho} E \left[ \frac{dn_t}{n_t} \right] / dt + \frac{x_t\nu_t(x_t\nu_t + \sigma_t^p + \sigma_t^K)^T}{\rho} \]

\[ - \frac{|\sigma_t^p + \sigma_t^K|^2}{2\rho} - \frac{x_t\nu_t(\sigma_t^p + \sigma_t^K)^T}{\rho} - \frac{x_t^2\nu_t^2}{2\rho} + H'_0(\eta_t)\mu_n^\eta \eta + \frac{H''_0(\eta_t)\eta^2\sigma_n^\eta(\sigma_n^\eta)^T}{2}. \]

Subtracting (5.19), we find that the difference in the utilities of these two agents, \( h(\eta_t) = H(\eta_t) - H_0(\eta_t) \), satisfies the ordinary differential equation

\[ \rho h(\eta_t) = \frac{x_t^2\nu_t\nu_t^T}{2\rho} + h'(\eta_t)\mu_n^\eta \eta + \frac{h''(\eta_t)\eta^2\sigma_n^\eta(\sigma_n^\eta)^T}{2}. \] (5.20)

### 6 Conclusion

We consider an economy in which household entrepreneurs and intermediaries make investment decisions. Household entrepreneurs can invest only in a single real production technology at a time, while intermediaries have the expertise to invest in a number of projects. In equilibrium intermediaries take advantage of their expertise to diversify across several investment projects. They scale up their activity by issuing demand deposits, *inside money*.  

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Households hold this inside money in addition to outside money provided by the government. Intermediaries are leveraged and assume liquidity mismatch. Intermediaries’ assets are long-dated and have low market liquidity - after an adverse shock the price can drop - while their debt financing is short-term. Endogenous risk emerges through amplification mechanism in form of two spirals. First, the liquidity spiral: a shock to intermediaries causes them to shrink balance sheets and “fire sale some of their assets”. This depresses the price of their assets which induces further fire-sales and so on. Second, the disinflationary spiral: as intermediaries shrink their balance sheet, they also create less inside money; such a shock leads to a rising demand for outside money, i.e. disinflation. This disinflationary spiral amplifies shocks, as it hurts borrowers who owe nominal debt. It works on the liabilities side of the intermediary balance sheets, while the liquidity spiral that hurts the price of capital works on the asset side. Importantly, in this economy the money multiplier, the ratio between inside and outside money, is endogenous: it depends on the health of the intermediary sector.

Monetary policy can mitigate the adverse effects of both spirals in the presence of default-free long-term government bonds. Conventional monetary policy changes the path of interest rate earned on short-term money and consequently impacts the relative value of long-term government bond and short-term money. For example, interest rate cuts in downturns that are expected to persist for a while enable intermediaries to refinance their long-bond holding more cheaply. This recapitalizes institutions that hold these assets and also increases the (nominal) supply of the safe asset. The resulting reduction in endogenous risk leads to welfare improvements. Of course, any policy that provides insurance against downturns could potentially create moral hazard. Indeed, intermediaries take on higher leverage, but more hazard is limited. The reason is that the “stealth recapitalization” through a persistent interest rate cut not only recapitalizes institutions with high leverage because they funded many real projects but also the ones which simply held long-term (default-free) Government bonds. The finding that moral hazard is limited might change if one were to include intermediaries with negative net worth. Including zombie banks is one fruitful direction to push this line of research further.

7 Bibliography


Bernanke, B. and A. Blinder (1989) “Credit, Money, and Aggregate Demand”, *American


A Numerical Procedure to find Equilibrium

(to be completed)

Computational Procedure. The function \( \vartheta(\eta) \) can be determined by a second-order differential equation, and the following procedure provides a way to find \( \vartheta''(\eta) \) from \( (\eta, \vartheta(\eta), \vartheta'(\eta)) \).

First, we have to guess variables \( \psi \) and \( \chi \) characterizing the allocation of capital and equity issuance, which satisfy the following conditions. Given \( \psi \) and \( \chi \), letting 
\[
\sigma = \sigma(\eta) = \frac{\sigma_a(1 - \psi) - \sigma_b(1 - \chi)}{\chi \eta - (1 - \psi)(1 - \chi) - \chi \eta}
\]
Define \( x = \psi(1 - \chi)/\eta \), and solve for household leverage \( x^a_t \) and \( x^b_t \) from the relative pricing and indifference conditions
\[
\frac{A^a(\psi_t)}{q_t} - \frac{\chi^\vartheta}{1 - \vartheta} w^2 = w \sigma^\kappa - \chi^\beta w^2
\]
\[
+(1 - \chi_t)(x_t(x_t^\varrho + \frac{\vartheta'}{\eta}\chi^\beta x_t)w^2 + x_t^a((\chi^\nu)^2 w^2 + \tilde{\sigma}^2) - x_t^b((\chi^\nu b)^2 w^2 + \tilde{\sigma}^2)
\]

and 
\[
(x_t^a)^2((\chi^\nu)^2 w^2 + \tilde{\sigma}^2) = (x_t^b)^2((\chi^\nu b)^2 w^2 + \tilde{\sigma}^2).
\]

The guesses of \( \psi \) and \( \chi \) are correct if
\[
\chi^\varrho(x_t^a + \frac{\vartheta'}{\eta}\chi^\beta)w^2 \geq x_t^a((\chi^\nu)^2 w^2 + \tilde{\sigma}^2), \quad \text{with strict inequality if } \chi_t = \chi
\]
and 
\[
\frac{\psi_t x_t^a + 1 - \psi_t}{x_t^b} = \frac{1 - \eta_t}{1 - \vartheta_t}.
\]

Finally, find
\[ \mu^\eta = (1 - \vartheta) \left( \frac{A_t(\psi_t) - \mu_t}{q_t} + \frac{(\chi^\eta)^2 w^2}{1 - \vartheta} - \frac{\sigma^2_t \chi^\eta}{1 - \vartheta} + \chi^\beta (x_t x^\nu + \frac{\vartheta}{\eta} \chi^\beta) w^2 + \right) \]

\[ \left( \psi^b - 1 - \frac{\chi^\eta}{1 - \vartheta} \right) \left( \chi^\beta w^2 - w \sigma^K \right) - x_t^b ((\chi^\nu x^b) w^2 + \tilde{\sigma}^2) \]

\[ \mu^\eta \eta = \eta (1 - \eta) \left( (x_t x^\nu + \frac{\vartheta}{\eta} \chi^\beta) w^2 - (x_t^b)^2 ((\chi^\nu x^b) w^2 + \tilde{\sigma}^2) \right) + \chi^\eta (\chi^\sigma - \chi^\beta) w^2 \]

and \[ \vartheta''(\eta) = \frac{2(\mu^\eta \vartheta(\eta) - \mu^\eta \eta \vartheta'(\eta))}{(\chi^\eta)^2 w^2}. \]

**Procedure.** The function \( \vartheta(\eta) \) that results under any policy that removes endogenous risk according to (5.13) has to satisfy the following second-order differential equation. First, given \( \eta \) and (\( \vartheta(\eta), \vartheta'(\eta) \)) the allocation of capital (\( \psi_t, \psi_t^J \)) must satisfy the conditions

\[ x_t = \frac{\psi_t (1 - \vartheta(\eta))}{\eta_t}, \quad \sigma^\eta_t = \frac{x_t \left( \sigma^J_t T - \sigma^K_t \right)}{1 - \frac{1 - \eta_t}{1 - \vartheta_t} \vartheta'(\eta)}, \quad \sigma^\vartheta_t = \frac{\vartheta'(\eta)}{\vartheta(\eta)} \sigma^\eta_t \eta_t, \quad Y - v(q) = \frac{\rho q}{1 - \vartheta_t}. \]

\[ x_t |\nu_t|^2 + \nu_t \frac{\vartheta_t (\sigma^\eta)^T}{\eta_t 1 - \vartheta_t} = x_t^J |\nu_t^J|^2, \quad \frac{\left( P_t^J - P_t^J \right) a}{q_t} = (\sigma^J 1^J - \sigma^J 1^J)^T (\sigma^K)^T + x_t^J |\nu_t^J|^2 - x_t^J |\nu_t^J|^2. \quad (A.1) \]

\[ \eta + \frac{\psi^J (1 - \vartheta(\eta))}{x_t^J \eta^J} + \frac{(1 - \psi^J - \psi) (1 - \vartheta(\eta))}{x_t^J \eta^J} = 1 \quad \text{and} \quad (x_t^J)^2 |\nu_t^J|^2 \leq (x_t^J)^2 |\nu_t^J|^2, \]

with equality of \( \psi + \psi^J < 1. \) Then,

\[ \frac{P_t^J a_t}{q_t} \sigma^\eta_t + \frac{\sigma^\eta_t}{1 - \vartheta_t} \sigma^\eta_t = x_t \left( \sigma^\eta_t + \frac{\vartheta_t \sigma^\eta_t}{1 - \vartheta_t} + \sigma^K_t \right)^T + \frac{\mu^\eta}{1 - \vartheta_t} \quad (A.2) \]

and \[ \vartheta''(\eta) = \frac{\mu^\eta \vartheta(\eta) - \vartheta'(\eta) \mu^\eta \eta}{\eta^2 \sigma^\eta_t (\sigma^\eta)^T / 2}, \]

where the volatility of \( \eta, \mu^\eta, \) is taken from (5.14).

**Proof.** We only need to justify the expressions that appear in the above procedure for the
first time. Note that

\[
\sigma^N_t = \sigma^K_t + \sigma^p_t - \frac{b_t}{pt}\sigma^B_t + x_t \left( \sigma^I_t \frac{1}{I} - \sigma^K_t \right) + \frac{\vartheta_t}{\eta_t} \sigma^\vartheta_t,
\]

where \( \sigma^N_t = \sigma^K_t + \sigma^q_t \).

Subtracting the capital pricing conditions for intermediaries and households who hold capital types \( i = 1, \ldots I \), we obtain

\[
x_t |\nu_t|^2 + \nu_t \frac{\vartheta_t (\sigma^\vartheta_t)^T}{\eta_t} = x'_t |\nu'_t|^2,
\]

since \( \nu_t (\sigma^M_t)^T = \nu'_t (\sigma^M_t)^T \). Furthermore, the capital pricing conditions of intermediaries and households who hold capital types \( j = I + 1, \ldots I + J \), we find

\[
\frac{(P^i_t - P^j_t)a}{q_t} + (\sigma^I_t 1^i - \sigma^J_t 1^j)(\sigma^q_t)^T = x_t |\nu_t|^2 - x'_t |\nu'_t|^2 + (\sigma^I_t 1^i - \sigma^J_t 1^j)(\sigma^K_t + \sigma^q_t)^T,
\]

which simplifies to the second equation in (A.1).

Finally, from the capital and bond pricing conditions for intermediaries, we obtain

\[
E[dr^I_t - dr^M_t] = \frac{P^I_t a - \iota}{q} + \mu^q_t - \mu^p_t + \sigma^q_t \left( \sigma^I_t \frac{1}{I} \right)^T - \sigma^K_t (\sigma^K_t)^T + \frac{\sigma^\vartheta_t}{1 - \vartheta} (\sigma^N_t)^T = \nu_t \sigma^N_t
\]

Since

\[
\sigma^N_t = \sigma^\eta_t + \frac{\nu_t \sigma^\vartheta_t}{1 - \vartheta} + \sigma^M_t \quad \text{and}
\]

\[
\mu^p_t - \mu^q_t = \frac{\sigma^\vartheta_t}{1 - \vartheta} \sigma^p_t - \frac{(\sigma^\vartheta_t)^2}{1 - \vartheta} + \frac{\mu^\vartheta}{1 - \vartheta}, \quad \sigma^K_t - \sigma^\vartheta_t = \frac{\sigma^\vartheta_t}{1 - \vartheta},
\]

we have

\[
\frac{P^I_t a - \iota}{q} + \sigma^\vartheta_t \frac{\sigma^\eta_t}{1 - \vartheta} = \nu_t \left( \sigma^\eta_t + \frac{\vartheta_t \sigma^\vartheta_t}{1 - \vartheta} + \sigma^K_t \right)^T + \frac{\mu^\vartheta}{1 - \vartheta},
\]

as required. \qed

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B Proofs

Proof of Proposition 1. Consider the law of motion of net worth

\[ \frac{dn_t}{n_t} = \mu_t^n dt + \sigma_t^n dZ_t = dr_t^M - \rho dt + \left\{ x_t^a (\nu_t^a)^T ((x_t^a \nu_t^a + \sigma_t^M) dt + dZ_t) + x_t^b \tilde{\sigma}^a (x_t^b \tilde{\sigma}^a dt + d\tilde{Z}_t) \right\} \]

\[ \left\{ x_t^b (\nu_t^b)^T ((x_t^b \nu_t^b + \sigma_t^M) dt + dZ_t) + x_t^b \tilde{\sigma}^b (x_t^b \tilde{\sigma}^b dt + d\tilde{Z}_t) \right\}, \]

depending on whether the household employs technology \( a \) or \( b \).

According to (3.6), the household gets the same utility from any choice over these two technologies if and only if \( \mu_t^n - |\sigma_t^n|^2 / 2 \) is the same for both technologies. For technology \( a \), this is

\[ E[dr_t^M] dt - \rho + (x_t^a)^2 (|\nu_t^a|^2 + (\tilde{\sigma}^a)^2) + x_t^a (\nu_t^a)^T \sigma_t^M - \frac{|x_t^a \nu_t^a + \sigma_t^M|^2 + (x_t^a \tilde{\sigma}^a)^2}{2} = \]

\[ E[dr_t^M] dt - \rho + \frac{(x_t^a)^2 (|\nu_t^a|^2 + (\tilde{\sigma}^a)^2) - |\sigma_t^M|^2}{2}. \]

Equating this for technologies \( a \) and \( b \), we obtain the indifference condition (2.12).

Lemma 1. Suppose that \( \eta = 0 \), i.e. there are no intermediaries. The equilibrium is characterized by a single equation for the allocation \( \psi \) of capital to technology \( b \)

\[ \frac{A_t^a(\psi) - A_t^b(\psi)}{q} = \frac{\rho}{x^a} - \frac{\rho}{x^b} + (1 - \psi)(\sigma^a)^2 - \psi(\sigma^b)^2, \quad (B.1) \]

with the remaining quantities expressed as

\[ x^a = \sqrt{\frac{\rho}{\psi^2((\sigma^a)^2 + (\sigma^b)^2) + (\tilde{\sigma}^a)^2}}, \quad x^b = \sqrt{\frac{\rho}{(1 - \psi)^2((\sigma^a)^2 + (\sigma^b)^2) + (\tilde{\sigma}^b)^2}}, \quad (B.2) \]

\[ 1 - \psi = \frac{x^a x^b}{(1 - \psi)x^b + \psi x^a} \quad \frac{\rho q}{1 - \psi} = A(\psi) - t(q) \quad \text{and} \quad p = \frac{\psi}{1 - \psi} q. \quad (B.3) \]

Proof. The aggregate risk of capital is \( \sigma^K dZ_t = (1 - \psi)\sigma^a 1^a dZ_t^a + \psi \sigma^b 1^b dZ_t^b \), incremental aggregate risks from exposures to technologies \( a \) and \( b \) are

\[ \nu_t^a = \psi (1^a \sigma^a - 1^b \sigma^b) dZ_t, \quad \nu_t^b = (1 - \psi) (1^b \sigma^b - 1^a \sigma^a) dZ_t, \]

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and the risk exposure of households in sectors a and b are

\[ x^a \nu_t^a \, dZ_t + \sigma^K \, dZ_t + x^a \tilde{\sigma}^a \, d\tilde{Z}_t \quad \text{and} \quad x^b \nu_t^b \, dZ_t + \sigma^K \, dZ_t + x^b \tilde{\sigma}^b \, d\tilde{Z}_t, \]

respectively. Hence, household indifference condition is

\[ X \equiv (x^a)^2 \left( \frac{\psi^2 (\sigma^a)^2 + (\sigma^b)^2}{|\nu_t^a|^2} \right) \quad \text{and} \quad \left( \frac{\sigma^a}{\nu_t^a} \right)^2 \left( 1 - \psi \right) (\sigma^a)^2 - \psi (\sigma^b)^2. \]  

The asset-pricing conditions for capital employed in sectors a and b are

\[ \frac{A^a(\psi) - \iota}{q} = x^a (|\nu_t^a|^2 + (\tilde{\sigma}^a)^2) + \psi((1 - \psi)(\sigma^a)^2 - \psi (\sigma^b)^2) \quad \text{and} \]

\[ \frac{A^b(\psi) - \iota}{q} = x^b (|\nu_t^b|^2 + (\tilde{\sigma}^b)^2) + (1 - \psi)(\psi (\sigma^b)^2 - (1 - \psi)(\sigma^a)^2). \]

Adding up these two equations with coefficients \( 1 - \psi \) and \( \psi \), and using the market-clearing condition for capital, we obtain

\[ \frac{\rho}{1 - \vartheta} = \left( \frac{1 - \psi}{x^a} + \frac{\psi}{x^b} \right) X \Rightarrow X = \rho. \]

This, together with (B.4), implies (B.2).

Equations in (B.3) follow from (2.14), market clearing condition for output and the definition of \( \vartheta \).

Finally, the difference between the asset-pricing conditions is

\[ \frac{A^a(\psi) - A^b(\psi)}{q} = x^a (|\nu_t^a|^2 + (\tilde{\sigma}^a)^2) - x^b (|\nu_t^b|^2 + (\tilde{\sigma}^b)^2) + (1 - \psi)(\sigma^a)^2 - \psi (\sigma^b)^2. \]

This yields (B.1).

**Proof of Proposition 3.** Notice that

\[ n_s = n_t \exp \left( \int_t^s \left( \mu_s^n - \frac{|\sigma_s^n|^2}{2} \right) \, ds' + \int_t^s \sigma_{s'}^n \, dZ_t \right), \]

since Ito’s lemma implies that then process \( n_s \) satisfies (3.5) as required.
Hence,
\[ E_t[\log(\rho_{n_s})] = \log(\rho_{n_t}) + E_t \left[ \int_t^s \left( \mu_{n_s}^n - \frac{|\sigma_{n_s}^n|^2}{2} \right) ds' \right]. \]

Integrating over \([t, \infty)\) and discounting, we obtain
\[ E_t \left[ \int_t^\infty e^{-\rho(s-t)} \log(\rho_{n_s}) ds \right] = \frac{\log(\rho_{n_t})}{\rho} + E_t \left[ \int_t^\infty e^{-\rho(s-t)} \int_t^s \left( \mu_{n_s}^n - \frac{|\sigma_{n_s}^n|^2}{2} \right) ds' ds \right], \]

which yields (3.6) after changing the order of integration.

\[ \square \]

Proof of Proposition 5. Let us normalize \( K_0 = 1 \). Consider an economy, in which households are required to allocate fraction \( \vartheta \in [0, 1) \) of their wealth to money. Then, from the market-clearing condition for consumption goods, if \( \Phi(\iota) = \log(\kappa\iota + 1)/\kappa \), then
\[ \bar{A} - \iota(q) = \rho \left( p + q \right) \Rightarrow q = \frac{(\kappa\bar{A} + 1)(1 - \vartheta)}{\kappa\rho + 1 - \vartheta} \text{ and } \Phi(\iota) = \frac{\log(q)}{\kappa}. \]

Then (3.7) implies that, given policy,
\[ \mu^n = \Phi(\iota) - \delta, \quad |\sigma^n|^2 = (1 - \vartheta)^2 \hat{\sigma}^2 + \bar{\sigma}^2, \]

hence welfare (3.6) is
\[ \frac{\log(\rho(p + q)) + \mu^n - |\sigma^n|^2/2}{\rho} = \frac{\log(\rho) + \log(q/(1 - \vartheta))}{\rho} + \frac{\log(q) + \kappa - \delta - (1 - \vartheta)^2 \hat{\sigma}^2/2 - \bar{\sigma}^2/2}{\rho^2} \]
\[ = \frac{\log(\rho)}{\rho} - \frac{\delta + \hat{\sigma}^2/2}{\rho^2} + \frac{\kappa + 1}{\kappa\rho} \left( \frac{\log(\kappa\bar{A} + 1)}{\rho} - \frac{\log(\kappa\rho + 1 - \vartheta)}{\rho} \right) + \frac{\log(1 - \vartheta) - (1 - \vartheta)^2 \hat{\sigma}^2}{2\rho^2}. \]

Let us show that welfare in the equilibrium with money is greater than in that without money. In the equilibrium with money \( 1 - \vartheta = \sqrt{\rho}/\hat{\sigma} \), so we need to show that
\[ -\frac{\kappa\rho + 1}{\kappa\rho} \log(\kappa\rho + \sqrt{\rho}/\hat{\sigma}) + \frac{1}{\kappa\rho} \log(\sqrt{\rho}/\hat{\sigma}) - \frac{1}{2} \geq -\frac{\kappa\rho + 1}{\kappa\rho} \log(\kappa\rho + 1) + \frac{\hat{\sigma}^2}{2\rho} \]
\[ \Rightarrow \]
\[ -\frac{\kappa\rho + 1}{\kappa\rho} \log(\kappa\rho x + 1) + \log x + \frac{x^2}{2} \geq -\frac{\kappa\rho + 1}{\kappa\rho} \log(\kappa\rho + 1) + \frac{1}{2} \]

(B.5)

where \( x = \hat{\sigma}/\sqrt{\rho} > 1 \). If we set \( x = 1 \), the two sides are equal. Differentiating the left-hand
side with respect to \(x\) we obtain
\[
\frac{1 - x}{(\kappa \rho x + 1)x} + x \geq \frac{1 - x}{x} + x = \frac{1 - x + x^2}{x} > 0.
\]
Hence (B.5) holds for all \(x > 1\), i.e. the equilibrium with money is strictly better.

Now, consider the optimal policy. Differentiating welfare with respect to \(\vartheta\) we get
\[
\frac{\kappa \rho + 1}{\kappa \rho^2} \frac{1}{\kappa \rho + 1 - \vartheta} - \frac{1}{\kappa \rho^2 (1 - \vartheta)} + \frac{(1 - \vartheta) \hat{\sigma}^2}{\rho^2} = \frac{-\vartheta}{\rho (\kappa \rho + 1 - \vartheta)} + \frac{(\vartheta - 1)^2 \hat{\sigma}^2}{\rho (1 - \vartheta)},
\]
where the term in parentheses is increasing in \(\vartheta\). For the equilibrium level of \(\vartheta = 1 - \sqrt{\rho/\hat{\sigma}}\), this term becomes
\[
\frac{\sqrt{\rho/\hat{\sigma}} - 1}{\kappa \rho + \sqrt{\rho/\hat{\sigma}}} + 1 = \frac{2 \sqrt{\rho/\hat{\sigma}} - 1 + \kappa \rho}{\kappa \rho + \sqrt{\rho/\hat{\sigma}}},
\]
positive if and only if \(2 \sqrt{\rho/\hat{\sigma}} > 1 - \kappa \rho\). Thus, the welfare-maximizing policy raises \(\vartheta\) over the equilibrium level if and only if condition (3.9) holds.

**Alternative Proof of Proposition 9.** Money has risk \(\sigma^M_t = \sigma^K_t + \sigma^p_t - b_t/p_t \sigma^B_t\). Effectively, risk \(b_t/p_t \sigma^B_t\) is subtracted from money and is allowed to be traded separately, carrying the risk premium of \(b_t/p_t(\sigma^B_t)^T \sigma^N_t\), since it is the intermediaries that hold bonds. The net worth of intermediaries follows
\[
\frac{dN_t}{N_t} = dr^M_t - \rho dt + (x_t \nu^B_t + x_t^B \sigma^B_t)^T (\sigma^N_t dt + dZ_t), \text{ where } x_t^B = \frac{b_t}{\eta_t (p_t + q_t)}.
\]

World wealth follows
\[
\frac{d((q_t + p_t) K_t)}{(q_t + p_t) K_t} = dr^M_t - \rho dt - (\sigma^g_t)^T (\sigma^M_t dt + dZ_t) + \eta_t (x_t^B \sigma^B_t)^T (\sigma^N_t dt + dZ_t),
\]
where we adjusted equation (2.17) for the presence of bonds in the world portfolio, as well as for the effect of bonds on the risk premium demanded by intermediaries.
Recall that risks that they take, must be Proof of Proposition 9. The law of motion of total net worth of intermediaries, given the

Therefore, using Ito’s lemma,

Therefore, the law of motion of aggregate wealth can be written as

The law of motion of world wealth \((q_t + p_t)K_t\) can be found from the total return on world wealth, after subtracting the dividend yield of \(\rho\) (i.e., aggregate consumption). To find the returns, we take into account the risk premia that various agents have to earn. We have

Recall that

Therefore, the law of motion of aggregate wealth can be written as
\((1 - \eta)(x_t^a)^2(|\nu_t^a|^2 + (\tilde{\sigma}^a)^2)\ dt + \eta|x_t^b + x_t^B\sigma_t^B|^2\ dt.\)

Thus, using Ito’s lemma, we obtain.\(^\text{15}\)

\[
\frac{d\eta_t}{\eta_t} = (x_t^b + x_t^B\sigma_t^B + \sigma_t^\theta - \eta x_t^B\sigma_t^B)^T(\sigma_t^M\ dt + \ dZ_t) + \\
(1 - \eta_t)|x_t^b + x_t^B\sigma_t^B|^2\ dt - (1 - \eta_t)(x_t^a)^2(|\nu_t^a|^2 + (\tilde{\sigma}^a)^2)\ dt
\]

\[-(x_t^b + x_t^B\sigma_t^B + \sigma_t^\theta - \eta x_t^B\sigma_t^B)^T(\sigma_t^M - \sigma_t^\theta + \eta x_t^B\sigma_t^B)\ dt = \\
(1 - \eta_t) ((x_t^b + x_t^B\sigma_t^B)^2 - (x_t^a)^2(|\nu_t^a|^2 + (\tilde{\sigma}^a)^2))\ dt + \\
(x_t^b + x_t^B\sigma_t^B + \sigma_t^\theta - \eta x_t^B\sigma_t^B)^T(dZ_t + (\sigma_t^\theta - \eta x_t^B\sigma_t^B)\ dt).
\]

Notice that

\[
\nu_t^b = 1^b\sigma^b + \sigma_t^\theta - \sigma_t^M = 1^b\sigma^b + \sigma_t^\theta - \sigma_t^\nu - \sigma_t^K + \frac{b_t}{p_t}\sigma_t^B = (1 - \psi_t)(1^b\sigma^b - 1^a\sigma^b) - \frac{\sigma_t^\theta}{1 - \vartheta} + \frac{b_t}{p_t}\sigma_t^B = \\
(1 - \psi_t)(1^b\sigma^b - 1^a\sigma^b) - \frac{\vartheta'(\eta)}{\vartheta(1 - \vartheta)}\eta\sigma_t^\theta + \frac{b_t}{p_t}\frac{B'(\eta)}{B}\eta\sigma_t^\theta.
\]

Hence, \(\sigma_t^\eta\) equals

\[
x_t(1 - \psi_t)(1^b\sigma^b - 1^a\sigma^b) - x_t \frac{\vartheta'(\eta)}{\vartheta(1 - \vartheta)}\eta\sigma_t^\theta + \frac{b_t}{p_t}\frac{B'(\eta)}{B}\eta\sigma_t^\theta = x_t(1 - \psi_t)(1^b\sigma^b - 1^a\sigma^b) - \left((1^b\psi_t - \eta_t) - (1 - \psi_t) + \vartheta_t(1 - \eta_t)\right)\frac{b_t}{p_t}\frac{B'(\eta)}{B}\sigma_t^\eta
\]

\(^\text{15}\)If processes \(X_t\) and \(Y_t\) follow

\[
dX_t/X_t = \mu_t^X\ dt + \sigma_t^X\ dZ_t \quad \text{and} \quad dY_t/Y_t = \mu_t^Y\ dt + \sigma_t^Y\ dZ_t,
\]

then

\[
\frac{d(X_t/Y_t)}{X_t/Y_t} = dX_t/X_t - dY_t/Y_t - (\sigma_t^X - \sigma_t^Y)^T\sigma_t^Y\ dt.
\]
It follows that
\[ \sigma_t^\eta = \frac{\chi_t (1 - \psi_t) (1^b \sigma^b - 1^a \sigma^a)}{1 + (\chi_t \psi_t - \eta_t) \partial_{\eta}^e - (\chi_t \psi_t (1 - \vartheta_t) + \vartheta_t (1 - \eta_t)) \frac{h_t}{B} \cdot \frac{B'}{B} \}. \]

**Proof of Proposition 10.** Equation (5.7) follows directly from Ito’s lemma. Let us justify the remaining six equations.

Relative to money, capital devoted to the production of good \( a \) earns the return of
\[ dr^a_t - dr^M_t = \frac{A^a(\psi_t) - \mu_t}{q_t} \, dt + (\mu^q_t - \mu^p_t) \, dt + (\sigma^a 1^a - \sigma^K_t) \, dZ_t + \sigma^a \, d\tilde{Z}_t. \]
Likewise,
\[ dr^b_t - dr^M_t = \frac{A^b(\psi_t) - \mu_t}{q_t} \, dt + (\mu^q_t - \mu^p_t) \, dt + (\sigma^b 1^b - \sigma^K_t) \, dZ_t + \sigma^b \, d\tilde{Z}_t. \]

Fraction \( 1 - \chi \) of the risk of good \( a \) is borne by intermediaries, who are exposed to aggregate risk \( \sigma^K_t \), and fraction \( \chi \), by households, who are exposed to aggregate risk \( \sigma^K_t \) and idiosyncratic risk \( \chi x_t^a \tilde{\sigma}^a \). Thus,
\[ \frac{A^a(\psi_t) - \mu_t}{q_t} + \mu^q_t - \mu^p_t = (\sigma^a 1^a - \sigma^K_t) \sigma^K + \chi x_t^a (\tilde{\sigma}^a)^2, \quad \text{(B.6)} \]
where \((\sigma^a 1^a - \sigma^K_t) \sigma^K \) is the risk premium for aggregate risk of this investment, and \( \chi x_t^a (\tilde{\sigma}^a)^2 \) is the price of idiosyncratic risk. For good \( b \),
\[ \frac{A^b(\psi_t) - \mu_t}{q_t} + \mu^q_t - \mu^p_t = (\sigma^b 1^b - \sigma^K_t) \sigma^K + \chi x_t^b (\tilde{\sigma}^b)^2. \quad \text{(B.7)} \]
Now, since any investment in capital will include a hedge for the aggregate risk component,
\[ \nu = \nu^b = 0, \]
so the indifference condition of households (2.12) becomes, one,
\[ (x_t^a)^2 (\tilde{\sigma}^a)^2 = (x_t^b)^2 (\tilde{\sigma}^b)^2 \iff x_t^a = x_t^b \frac{\tilde{\sigma}^b}{\tilde{\sigma}^a}. \]
and the law of motion of $\eta_t$ is, two,
\[
\frac{d\eta_t}{\eta_t} = -(1 - \eta_t)(x_t^b)^2(\bar{\sigma}^b)^2 \, dt.
\] (B.8)

From (2.14), we have, three,
\[
\frac{\psi_t \chi_t}{x_t^a} + \frac{1 - \psi_t}{x_t^b} = \frac{1 - \eta_t}{1 - \vartheta_t}.
\] (B.9)

Subtracting (B.7) from (B.6), we get, four,
\[
\frac{A^a(\psi_t) - A^b(\psi_t)}{q_t} = (\sigma^a 1^a - \sigma^b 1^b)^T \sigma^K + \chi x_t^a(\bar{\sigma}^a)^2 - x_t^b(\bar{\sigma}^b)^2.
\] (B.10)

The market-clearing condition for consumption goods is, five,
\[
A(\psi_t) - \iota(q_t) = \frac{\rho q_t}{1 - \vartheta_t}.
\]

Finally, taking a weighted average of (B.6) and (B.7), with weights $\psi$ and $1 - \psi$, we have
\[
\frac{A(\psi) - \iota_t}{q_t} + \mu_t^p - \mu_t^q = \frac{\chi \psi \bar{\sigma}^a / \bar{\sigma}^b + 1 - \psi}{\rho/(1 - \vartheta_t)} x_t^b(\bar{\sigma}^b)^2.
\]

This, in combination with (B.9), and the identity $\mu_t^q = (1 - \vartheta_t)(\mu_t^p - \mu_t^q) - \sigma^q \sigma^p + (\sigma^q)^2$, leads to the last equation, (5.8), six.

C “As If” Representative Agent Model

Salient features of the various equilibria discussed above can also be achieved through closely related representative agent economies. Suppose the representative agent is endowed with initial capital $K_0 \equiv \int_0^1 k_i \, di$ and that he faces technology-specific, but no purely idiosyncratic risk. Let us furthermore assume that this representative agent also has log preferences, but now with discount factor $\tilde{\rho}$. For starters, suppose that he can only invest in technologies $a$ or $b$ (no money is available). In that case his portfolio problem is simple: He will consume a constant fraction $\tilde{\rho}$ of his wealth, he will invest equal fractions of his capital at each instant.
in technologies \(a\) and \(b\), and the price of capital will be time-invariant and satisfy

\[ q^R = \frac{\kappa \bar{A} + 1}{\kappa \bar{\rho} + 1} \]

For \(\bar{\rho} = \rho\) we exactly recover the no-money equilibrium above, with the sole difference that welfare of the representative households exceeds welfare of the atomistic households above (since the representative household need not bear any idiosyncratic risk). A formal analysis of the welfare difference will follow in the next section. It is also instructive to consider the case \(\bar{\rho} = \sqrt{\rho \hat{\sigma}}\). In that case, the equilibrium with the representative agent is, as far as real quantities and the price of capital are concerned, exactly the same as the money equilibrium above. Since \(\bar{\rho} > \rho\), we see that the money equilibrium effectively amounts to an decrease in patience.

Finally, it is interesting to consider the question of whether money can have value in a representative agent economy. Evidently, the answer is no – an asset that never generates any real payoff cannot be held in positive quantities forever without violating the investor’s transversality condition. However, let us for the moment assume that the transversality condition is allowed to be violated. In that case, as long as the representative household has no purely idiosyncratic risk, there still could be no equilibrium with constant \(p, q \gg 0\), for money would be a strictly dominated asset. We could of course deal with strict dominance by re-introducing idiosyncratic risk, but then the aggregate law of motion for capital would be different (since now idiosyncratic cannot cancel in the aggregate). Only if the representative agent would perceive the threat of idiosyncratic risk at every period, but without this risk every materializing, would we recover allocations and prices from the money equilibrium above – and all of course subject to the proviso of ignoring the representative household’s transversality condition.

D Old Material

D.1 Welfare Analysis, No Intermediaries

What if the regulator can control \(\vartheta\) by forcing the agents to hold specific amounts of money? In this case, we have to use the economy-size method which directly takes into account the cost idiosyncratic risk exposure, without assuming that agents earn the required return for
the risk that they take.\footnote{When the regulator controls portfolios, it is no longer true that the excess return of any risky asset over any other asset is explained by their covariance with the agent’s net worth.} Since the choice of $\vartheta$ will affect asset prices, and thus wealth, we can calculate welfare per unit of capital in the economy to be

## D.2 OLD Welfare, with Intermediary Sector

**The Investment Return Method.** The following proposition evaluates the welfare of any agent as a function of his/her investment opportunities.

**Proposition 13.** \textit{The welfare of an agent with wealth $n_t$ who can invest only in money takes the form $h^M(\eta_t) + \log(\rho n_t)/\rho$, where $h^M(\eta_t)$ satisfies}

$$h^M(\eta_t) + \frac{\log p_t}{\rho} = E_t \left[ \int_t^\infty e^{-\rho(s-t)} \left( \log(p_s) + \frac{\Phi(t_s) - \delta - \rho - |\sigma^K_s|^2/2}{\rho} \right) ds \right] \quad \text{(D.1)}$$

\text{The welfare of an intermediary with net worth $n_t$ is $h^I(\eta_t) + \log(\rho n_t)/\rho$, where}

$$h^I(\eta_t) - h^M(\eta_t) = E_t \left[ \int_t^\infty e^{-\rho(s-t)} \frac{x^b_s|\nu^b_s|^2}{2\rho} ds \right] \quad \text{(D.2)}$$

\text{The welfare of a household is $h^H(\eta_t) + \log(\rho n_t)/\rho$, where $h^H(\eta_t) - h^M(\eta_t)$ satisfies equation (D.2) with the term $x^b_s|\nu^b_s|^2$ replaced by $(x^b_s)^2(|\nu^b_s|^2 + (\tilde{\sigma}^b)^2)$.

Equation (D.2) evaluates the welfare of an individual agent by inferring the Sharpe ratio that the agent earns on risky investment from the risk that the agent chooses to take. The expectations (D.1) and (D.2) can be found numerically through an ordinary differential equation, since

$$g(\eta_t) = E_t \left[ \int_t^\infty e^{-\rho(s-t)} y(\eta_s) ds \right] \Rightarrow \rho g(\eta) = y(\eta) + g'(\eta)\mu^\eta + \frac{g''(\eta)|\eta\sigma^\eta|^2}{2} \quad \text{(D.3)}$$

Throughout the proofs we also make use of the following identity

$$\rho g(\eta_t) = y(\eta_t) + E_t[dg(\eta_t)]/dt \Rightarrow g(\eta_t) = E_t \left[ \int_t^\infty e^{-\rho(s-t)} y(\eta_s) ds \right]. \quad \text{(D.4)}$$

Note that, given the form of equation (D.2), it makes sense why Proposition 1 gives the right condition for the household to be indifferent between the production of goods $a$ and $b$.}
Under condition (2.12), the household has the same welfare regardless of the technology it chooses to pursue.

**Proof.** With log utility, if the wealth of the agent increases by a factor of \( y \), then his/her utility increases by \( \log(y)/\rho \), since the agent increases consumption by a factor of \( y \) in perpetuity and keeps portfolio weights the same.

We can write the utility of an agent with wealth \( n_t \) who can invest only in money in the form \( h^M(\eta_t) + \log(p_t)/\rho + \log(n_t)/\rho \), and if we express \( n_t = p_t k_t \), then

\[
\frac{dk_t}{k_t} = (\Phi(t_t) - \delta - \rho) \, dt + \sigma^K_t \, dZ_t,
\]

since the agent consumes at rate \( \rho \). Then the agent’s utility has to satisfy the equation

\[
\rho \left( h^M(\eta_t) + \frac{\log(p_t)}{\rho} \right) = \log(p_t) + \frac{E_t}{dt} \left[ d \left( h^M(\eta_t) + \log(k_t)/\rho + \log(p_t)/\rho \right) \right] \Rightarrow
\]

\[
\rho \left( h^M(\eta_t) + \frac{\log(p_t)}{\rho} \right) = \log(p_t) + \frac{\Phi(t_t) - \delta - \rho}{\rho} - \frac{|\sigma^K_t|^2}{2\rho} + \frac{E_t}{dt} \left[ d \left( h^M(\eta_t) + \log(p_t)/\rho \right) \right]
\]

so (D.4) implies that \( h^M(\eta) + \log(p_t)/\rho \) satisfies (D.1).

The net worth of this agent follows

\[
\frac{dn_t}{n_t} = dr^M - \rho \, dt,
\]

and recall that the net worth of an intermediary follows (2.16), i.e.

\[
\frac{dn^I_t}{n^I_t} = dr^M - \rho \, dt + x_t(\nu^b_t)^T((x_t\nu^b_t + \sigma^M_t) \Sigma^N_t \, dt + dZ_t).
\]

If we write the utilities of these agents as \( h^M_t = \log(p_t)/\rho \) and \( h^I_t = \log(p^I_t)/\rho \) then we have

\[
\rho \left( h^M_t + \frac{\log(p_t)}{\rho} \right) = \log(p_t) + \frac{1}{\rho} E_t \left[ \frac{dn_t}{n_t} \right] /dt - \frac{|\sigma_t^M|^2}{2\rho} + E_t[dr^M_t]/dt \quad \text{and}
\]

\[
\rho \left( h^I_t + \frac{\log(p^I_t)}{\rho} \right) = \log(p^I_t) + \frac{1}{\rho} E_t \left[ \frac{dn^I_t}{n^I_t} \right] /dt - \frac{|\sigma_t^N|^2}{2\rho} + E_t[dr^I_t]/dt.
\]

Subtracting, we find that

\[
\rho(h^I_t - h^M_t) = \frac{2x_t(\nu^b_t)^T(x_t\nu^b_t + \sigma^M_t)}{2\rho} - \frac{|\sigma_t^N|^2 - |\sigma_t^M|^2}{2\rho} + E_t[dr^I_t - dr^M_t]/dt.
\]
It follows from (D.4) that \( h^I(\eta_t) - h^M(\eta_t) = h^I_t - h^M_t \) is represented by the stochastic expectation (D.2).

The logic for the characterization of the welfare of households is analogous. \( \square \)

**The Economy Size Method.** The following proposition provides another way to evaluate the welfare of intermediaries and households.

**Proposition 14.** The equilibrium utility of an intermediary with net worth \( n_t \) is

\[
h^I(\eta_t) = E_t \left[ \int_t^\infty e^{-\rho(s-t)} \left( \log(\eta_s(p_s + q_s)) + \frac{\Phi(\iota_s) - \delta}{\rho} - \frac{|\sigma^K_s|^2}{2\rho} \right) ds \right] - \frac{\log(\eta_t(p_t + q_t))}{\rho},
\]

where

\[
h^I(\eta_t) = E_t \left[ \int_t^\infty e^{-\rho(s-t)} \left( \log(\eta_s(p_s + q_s)) + \frac{\Phi(\iota_s) - \delta}{\rho} - \frac{|\sigma^K_s|^2}{2\rho} \right) ds \right] - \frac{\log((1-\eta_s)(p_s + q_s))}{\rho}.
\]

The equilibrium utility of a household is

\[
h^H(\eta_t) = E_t \left[ \int_t^\infty e^{-\rho(s-t)} \left( \log((1-\eta_s)(p_s + q_s)) + \frac{\Phi(\iota_s) - \delta}{\rho} - \frac{|\sigma^K_s|^2}{2\rho} \right) ds \right] - \frac{\log((1-\eta_t)(p_t + q_t))}{\rho} + \frac{1}{2\rho} E_t \left[ \int_t^\infty e^{-\rho(s-t)} \left( \frac{\eta_s \sigma^\eta_s}{1-\eta_s} + \sigma^\iota_t \right)^2 - (\nu^b_s)^2 (|\nu^b_s|^2 + (\bar{\sigma}^b)^2) ds \right].
\]

The intuition behind equations (D.5) and (D.6) is as follows. Note that an intermediary with a unit net worth at time \( t \) will have the net worth of

\[
\frac{\eta_s(p_s + q_s)K_s}{\eta_t(p_t + q_t)K_t}
\]

at time \( s \geq t \) and will consume \( \rho \) times net worth. The utility of consumption is

\[
\log(\rho \eta_s(p_s + q_s)) - \log(\eta_t(p_t + q_t)) + \log \frac{K_s}{K_t},
\]

and equation (D.5) reflects exactly that: the utility of an intermediary through the evolution of \( \eta_t \) and world capital.

Equation (D.6) follows the same logic, but adjusts for the risk that individual households take - including idiosyncratic risk - relative to the risk of \( 1 - \eta_t \). Note that from (D.6), it is also clear why the condition of Proposition 1 is the right condition for households to be indifferent between producing goods \( a \) and \( b \).
Proof. Consider an intermediary with net worth $n_t = y n_t (p_t + q_t) K_t$. The intermediary will consume $\rho y n_t (p_t + q_t) K_t$, so

$$\rho \left( h^I(n_t) + \frac{\log(\rho y n_t (p_t + q_t) K_t)}{\rho} \right) =$$

$$\log(\rho y n_t (p_t + q_t) K_t) + \frac{E_t \left[ d \left( h^I(n_t) + \log(n_t (p_t + q_t)) / \rho + \log(K_t) / \rho \right) \right]}{dt} \Rightarrow$$

$$\rho \left( h^I(n_t) + \frac{\log(n_t (p_t + q_t))}{\rho} \right) =$$

$$\log(n_t (p_t + q_t)) + \frac{\Phi(t_{t-}) - \delta}{\rho} - \frac{1}{2\rho} + \frac{E_t \left[ d \left( h^I(n_t) + \log(n_t (p_t + q_t)) / \rho \right) \right]}{dt} \Rightarrow$$

$$h^I(n_t) + \frac{\log(n_t (p_t + q_t))}{\rho} = E_t \left[ \int_{t}^{\infty} e^{-\rho(s-t)} \left( \log(n_s (p_s + q_s)) + \frac{\Phi(t_s) - \delta}{\rho} - \frac{1}{2\rho} \right) ds \right],$$

where the last step follows from (D.4). The last equation implies (D.5).

Now, consider a household with net worth $y(1 - n_t)(p_t + q_t) K_t$. If the net worth of this agent had evolved like the total net worth of all households, i.e.

$$\frac{dN^H_t}{N^H_t} = dr^M_t - \rho dt + \frac{(1 - \vartheta_t)(1 - \chi_t)}{1 - n_t}((\nu_t^b)^T (\sigma_t^{N_b} dt + dZ_t) + x_t^b (\sigma_b)^2 dt)

+ \frac{(1 - \vartheta_t)(1 - \psi_t)}{1 - n_t}((\nu_t^a)^T (\sigma_t^{N_a} dt + dZ_t) + x_t^a (\sigma_a)^2 dt),$$

then, likewise, the utility of this agent would be $h^{1-n}(n_t) + \log(\rho n_t) / \rho$, where

$$h^{1-n}(n_t) = E_t \left[ \int_{t}^{\infty} e^{-\rho(s-t)} \left( \log((1 - n_s) p_s / \vartheta_s) + \frac{\Phi(t_s) - \delta}{\rho} - \frac{1}{2\rho} \right) ds \right] - \log((1 - n_t) p_t / \vartheta_t).$$

Now, of course the net worth of a household that specializes in good $b$ follow instead

$$\frac{dn^b_t}{n^b_t} = dr^M_t - \rho dt + x_t^b ((\nu_t^b)^T (\sigma_t^{N_b} dt + dZ_t) + x_t^b (\sigma_b)^2 dt + \sigma_b^2 d\tilde{Z}_t).$$

By (2.14),

$$x_t^b - \frac{(1 - \vartheta_t)(1 - \chi_t)}{1 - n_t} = \frac{(1 - \vartheta_t)(1 - \psi_t)}{1 - n_t} \frac{x_t^b}{\tilde{x}_t^b}. \quad \text{(D.7)}$$
Thus, the difference in drifts of \( n_t^b \) and \( N_t^H \) is
\[
D'_t = \frac{(1 - \vartheta_t)(1 - \psi_t)}{1 - \eta_t} \frac{x_t^b}{x_t^b}((\nu_t^b)^T \sigma_t^N b + x_t^b(\bar{b}^b)^2) - \frac{(1 - \vartheta_t)(1 - \psi_t)}{1 - \eta_t} x_t^a ((\nu_t^a)^T (x_t^a \nu_t^a + \sigma_t^M) + x_t^a(\bar{a}^a)^2)
\]
by Proposition 1.

The difference between squared volatilities of \( n_t^b \) and \( N_t^H \) is
\[
D'^{\sigma}_t = \left| \sigma_t^M + x_t^b \nu_t^b \right|^2 + (x_t^b)^2(\bar{b}^b)^2 - \left| \sigma_t^M + \frac{(1 - \vartheta_t) \psi_t (1 - x_t)}{1 - \eta_t} \nu_t^b + \frac{(1 - \vartheta_t) (1 - \psi_t)}{1 - \eta_t} \nu_t^a \right|^2 =
\]
\[
2x_t^b(\nu_t^b)^T \sigma_t^M + (\nu_t^b)^2 + (x_t^b)^2(\bar{b}^b)^2 - 2 \left( \frac{(1 - \vartheta_t) \psi_t (1 - x_t)}{1 - \eta_t} \nu_t^b + \frac{(1 - \vartheta_t) (1 - \psi_t)}{1 - \eta_t} \nu_t^a \right)^T \sigma_t^M
\]
\[- \left| \frac{\eta_t \sigma_t^b}{1 - \eta} + \sigma_t^\theta \right|^2 = 2D'^{\theta}_t + (x_t^b)^2(\nu_t^b)^2 + (\bar{b}^b)^2 - \left| \frac{\eta_t \sigma_t^b}{1 - \eta} + \sigma_t^\theta \right|^2,
\]
where we used (D.7) and the fact that
\[
(1 - \vartheta_t)(1 - \psi_t) \nu_t^a + \psi_t \nu_t^b = -\sigma_t^\theta \quad \Rightarrow
\]
\[
\frac{(1 - \vartheta_t)(1 - \psi_t)}{1 - \eta_t} \nu_t^a + \frac{(1 - \vartheta_t) \psi_t (1 - x_t)}{1 - \eta_t} \nu_t^b = \frac{-\sigma_t^\theta}{1 - \eta} - \frac{(1 - \vartheta_t) \psi_t x_t \nu_t^b}{1 - \eta_t} - \frac{\eta_t \sigma_t^b}{1 - \eta_t} - \sigma_t^\theta.
\]

Thus,
\[
D'^{\theta}_s - D'^{\sigma}_s / 2 = \frac{1}{2} \left( \left| \frac{\eta_t \sigma_t^b}{1 - \eta_t} + \sigma_t^\theta \right|^2 - (x_t^b)^2(\nu_t^b)^2 + (\bar{b}^b)^2 \right),
\]
and
\[
h^H_t(\eta_t) - h^{1 - \eta}(\eta_t) = E_t \left[ \frac{1}{2 \rho} \int_0^\infty e^{-r(s-t)} \left( \left| \frac{\eta_t \sigma_t^b}{1 - \eta_t} + \sigma_t^\theta \right|^2 - (x_s^b)^2(\nu_s^b)^2 + (\bar{b}^b)^2 \right) ds \right].
\]

This completes the proof. \( \square \)